4.1: Exponential Functions

An *exponential* function takes the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

Example 1:

$f(x) = 2^x$

or $g(x) = 7^x$ or $h(x) = \left(\frac{2}{3}\right)^x$

Why must we have $b > 0$ and $b \neq 1$?

why $b > 0$?

Suppose $b = -4$ | $f(x) = (-4)^x$

What if $x = \frac{1}{2}$? $f(\frac{1}{2}) = (-4)^{1/2}$

$= \sqrt{-4}$ not a real number

why $b \neq 1$?

If $b = 1$, then

$f(x) = b^x$

$= 1^x = 1$

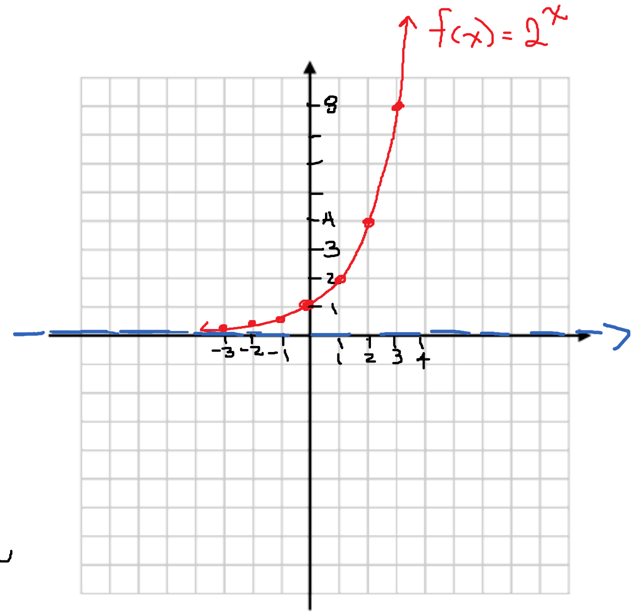
$f(x) = 1$

constant function

The graph of $f(x) = b^x$:

Example 2: Sketch the graph of $f(x) = 2^x$ by plotting points.

x	$f(x) = 2^x$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



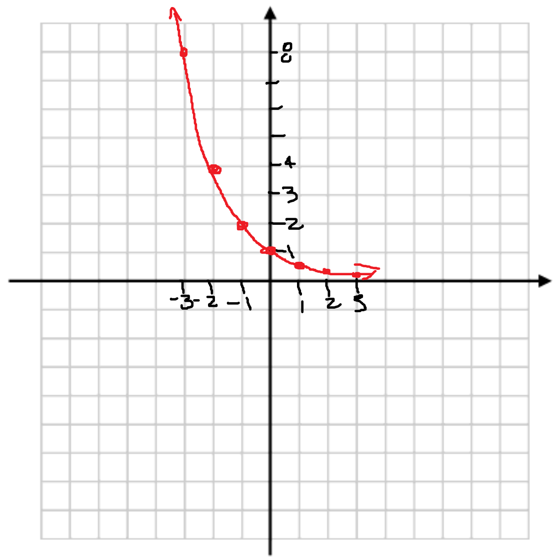
The x-axis is a horizontal asymptote for the graph

Horizontal asymptote: x-axis

$y = 0$

Example 3: Sketch the graph of $f(x)=\left(\frac{1}{2}\right)^x$ by plotting points.

x	$f(x)=\left(\frac{1}{2}\right)^x$
-3	$\left(\frac{1}{2}\right)^{-3} = \left(\frac{2}{1}\right)^3 = 2^3 = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = \frac{1^{-2}}{2^{-2}} = \frac{2^2}{1^2} = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = \left(\frac{2}{1}\right)^1 = 2^1 = 2$
0	$\left(\frac{1}{2}\right)^0 = 1$
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

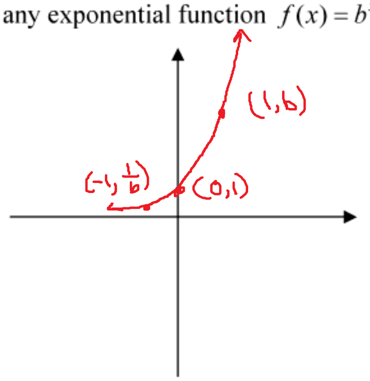


Note: $f(x)=\left(\frac{1}{2}\right)^x$ is the y-axis reflection of $y=2^x$

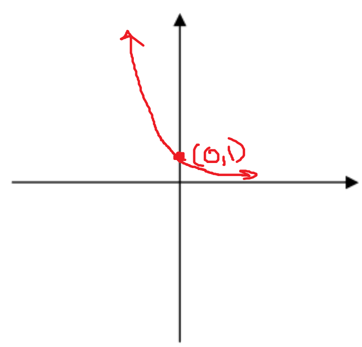
why? $y=\left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = \frac{2^{-x}}{1} = 2^{-x}$

Recall: replacing x by $-x$ reflects the graph around the y-axis

For any exponential function $f(x)=b^x$, the graph looks like one of the following.



$b>1$



$0<b<1$

Notice:
- Domain is $(-\infty,\infty)$.
- Range is $(0,\infty)$.
- Horizontal asymptote is $y=0$ (the x-axis).
- Always passes through the points $(-1,\frac{1}{b})$, $(0,1)$, $(1,b)$.

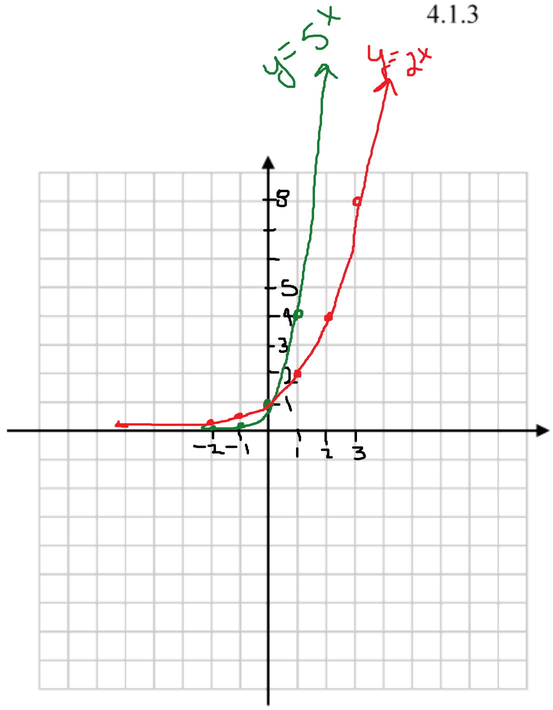
How do different bases affect the graph?

For $b > 1$, a larger b results in a steeper graph.

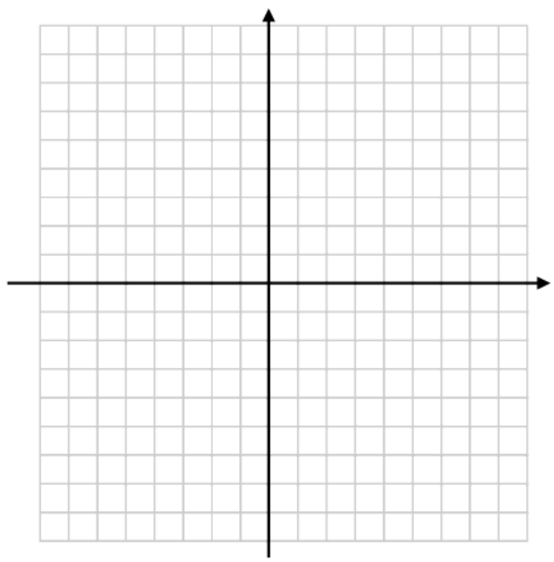
$y = 2^x$ $\qquad$ $y = 5^x$

x	$y = 2^x$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

x	$y = 5^x$
-2	$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
-1	$5^{-1} = \frac{1}{5}$
0	$5^0 = 1$
1	$5^1 = 5$
2	$5^2 = 25$



For $b < 1$, a smaller b results in a steeper graph.



Example 4: Sketch the graph of $y = 2 + 3^x$.

$y = 2 + 3^x$

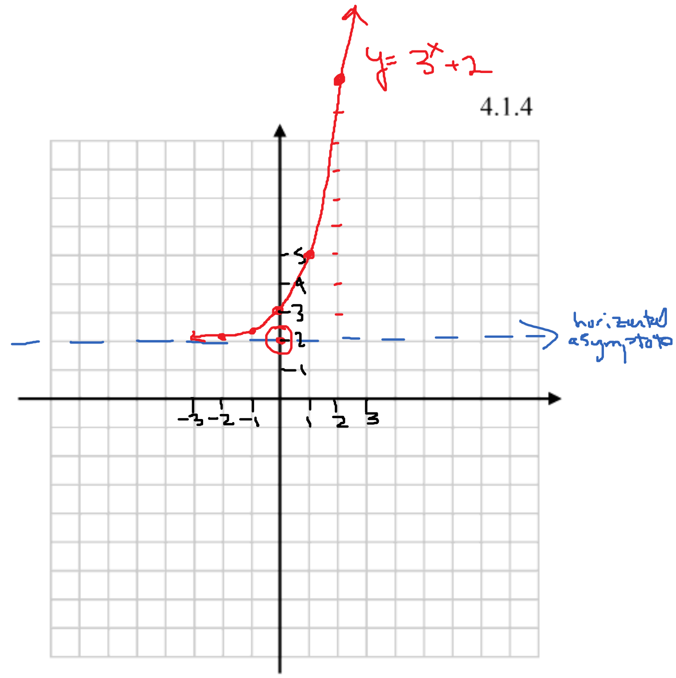
$y = 3^x + 2$

Parent function: $y = 3^x$

start with $y = 3^x$, shift it up 2

Parent Function

x	$y = 3^x$
-2	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
-1	$3^{-1} = \frac{1}{3}$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$



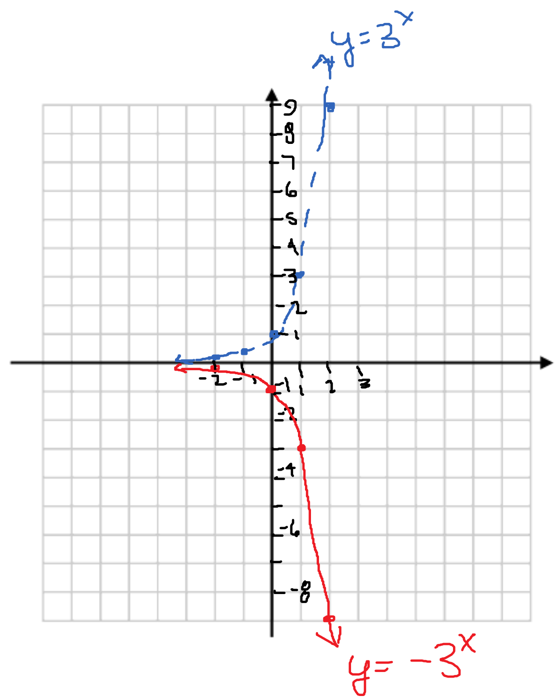
Example 5: Sketch the graph of $f(x) = -3^x$.

$y = -3^x$

Start with parent function $y = 3^x$

then reflect around x-axis

Same table of values as previous example



Example 6: Sketch the graph of $f(x) = 4^{x+1} - 3$.

$y = 4^{x+1} - 3$

Start with parent function $y = 4^x$

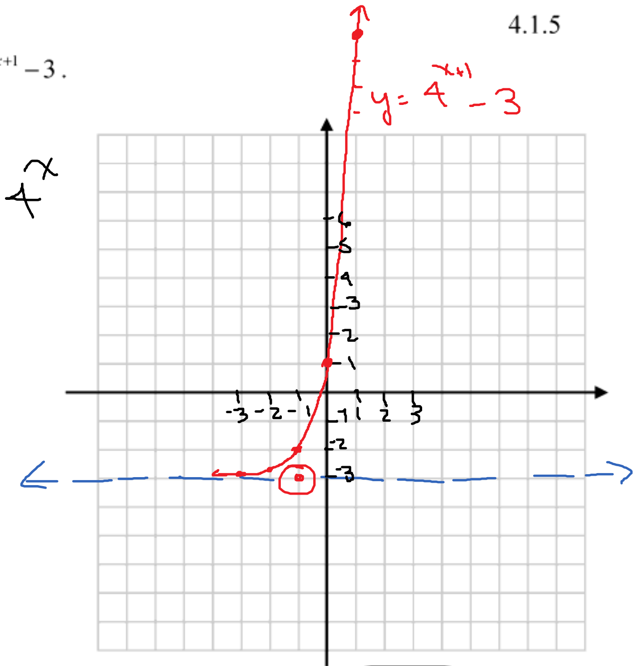
x	$y = 4^x$
-2	$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$
-1	$4^{-1} = \frac{1}{4}$
0	$4^0 = 1$
1	$4^1 = 4$
2	$4^2 = 16$

Shift it left 1, down 3

$x + 1 = 0$

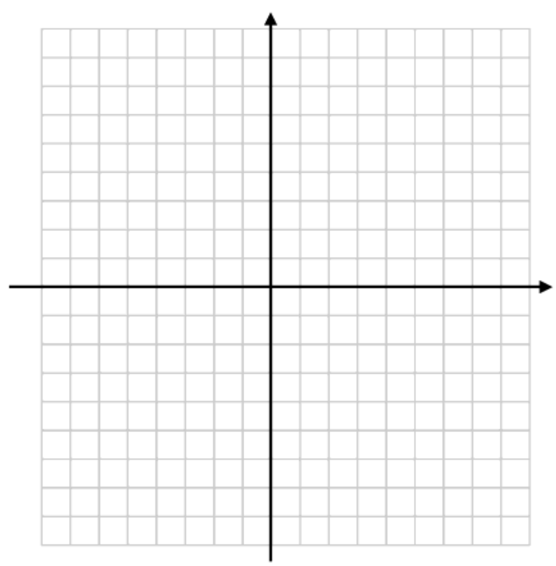
$x = -1$

shift left 1

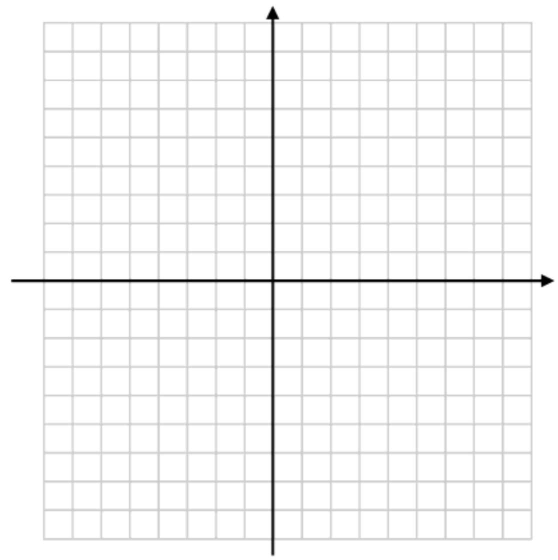


Horizontal asymptote: $y = -3$

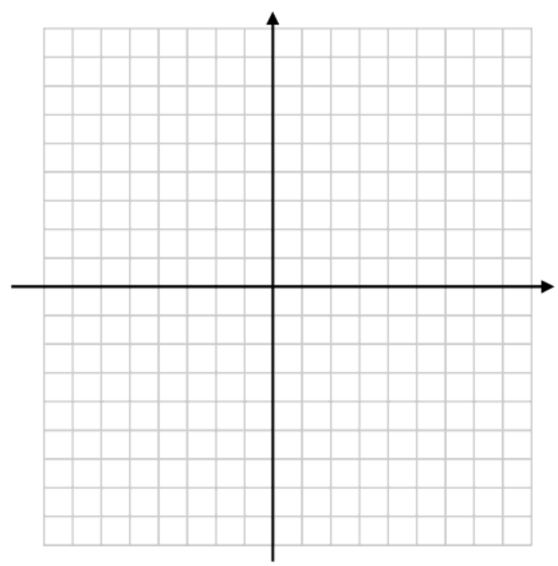
Example 7: Sketch the graph of $y = 2 \cdot 3^x$.



Example 8: Sketch the graph of $f(x)=\left(\frac{1}{2}\right)^{x+1}$.



Example 9: Sketch the graph of $f(x)=\left(\frac{2}{3}\right)^{x}$.



The Number *e*:

We've worked with 2^x, 3^x, etc. Now we have e^x.

What is *e*? *e* is a very important number. Definition: *e* is the "limiting value" of $(1+\frac{1}{x})^x$ as *x* grows to infinity.

$$\boxed{e \approx 2.718281828459}$$

It is an irrational number, like π. This means it cannot be written as a fraction nor as a terminating or repeating decimal. Unless otherwise asked, leave *e* and π as e and π! Do not approximate!

Remember: *e* is a number, just as 2, 3, and 17 are numbers. So it can be treated the same way.

In mathematics, it is very rare for anyone to use *e* as a variable.

Believe it or not: $f(x)=e^x$ is a much "nicer" function than $f(x)=2^x$. In fact, you must change 2^x to e^{cx} (*c* a number), before you can do any calculus on it.

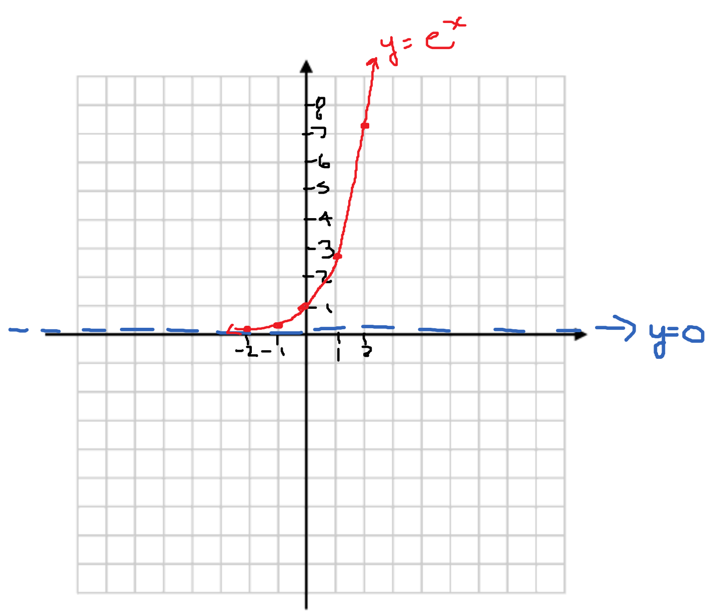
The *natural exponential function* is $f(x)=e^x$.

The graph of $f(x)=e^x$:

Since $e>1$, its graph looks like:

x	$y=e^x$
-2	$e^{-2}=\frac{1}{e^2} \approx \frac{1}{9}$
-1	$e^{-1}=\frac{1}{e} \approx \frac{1}{2.7} \approx \frac{1}{3}$
0	$e^0=1$
1	$e^1 \approx 2.7$
2	$e^2 \approx 7.29$

$$\begin{array}{r} {}^{4} \\ 2.7 \\ \times\ 2.7 \\ \hline 189 \\ 54\ \ \\ \hline 7.29 \end{array}$$

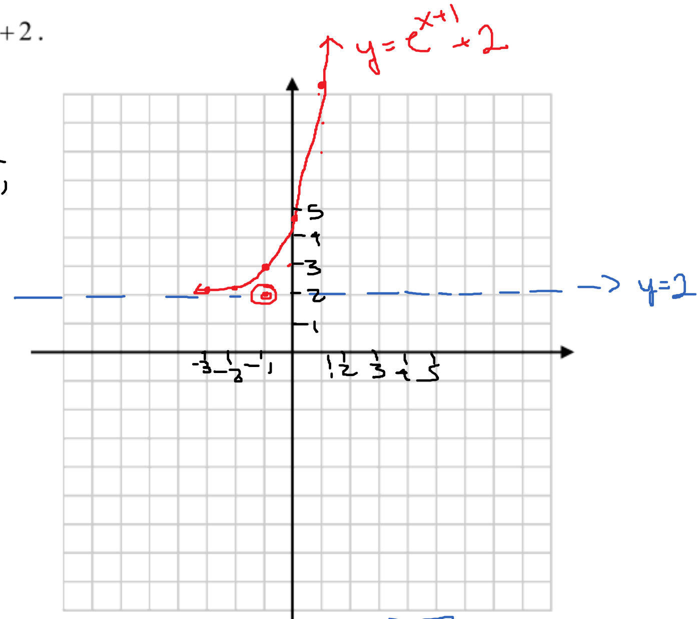


Example 10: Sketch the graph of $f(x) = e^{x+1} + 2$.

$y = e^{x+1} + 2$

Start with parent function $y = e^x$, then shift it left 1 and up 2

x	$y = e^x$
-2	$e^{-2} = \frac{1}{e^2} \approx \frac{1}{9}$
-1	$e^{-1} = \frac{1}{e} \approx \frac{1}{3}$
0	$e^0 = 1$
1	$e^1 = e \approx 2.7$
2	$e^2 \approx 7.3$



Eqn of asymptote: $\boxed{y = 2}$