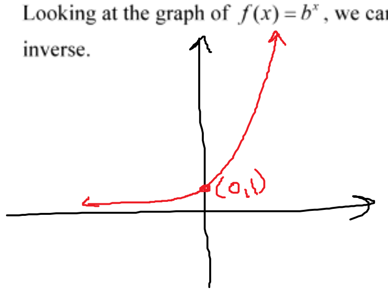
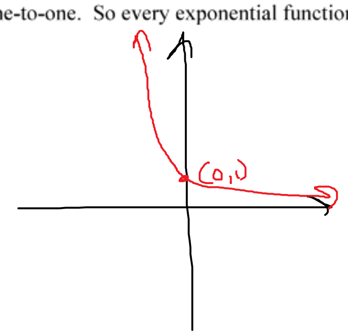
4.2: Logarithmic Functions

Looking at the graph of $f(x) = b^x$, we can see it is one-to-one. So every exponential function has an inverse.



$b > 1$



$0 < b < 1$

Horizontal Line test tells us these are 1-1. (they have inverses)

Example 1: Consider the function $f(x) = 3^x$.

Then $f(1) = 3$, $f(2) = 9$, $f(3) = 27$, and $f(4) = 81$. This function has an inverse.

So $f^{-1}(3) = 1$, $f^{-1}(9) = 2$, $f^{-1}(27) = 3$, and $f^{-1}(81) = 4$.

We call this inverse function a *logarithmic function* and denote it $f^{-1}(x) = \log_3 x$.

So $f^{-1}(3) = \log_3 3 = 1$ and $f^{-1}(9) = \log_3 9 = 2$. Also $\log_3 27 = 3$ and $\log_3 81 = 4$.

Note: Shorthand: $\log_3 x$ means $\log_3(x)$

x	$f(x) = 3^x$
0	$f(0) = 3^0 = 1$
1	$f(1) = 3^1 = 3$
2	$f(2) = 3^2 = 9$
3	$f(3)$ $3^3 = 27$
4	$f(4) = 3^4 = 81$

$\Longrightarrow$

x	$f^{-1}(x) = \log_3(x)$
1	$0 = f^{-1}(1) \Rightarrow \log_3(1) = 0$
3	$1 = f^{-1}(3) \Rightarrow \log_3(3) = 1$
9	$2 = f^{-1}(9) \Rightarrow \log_3(9) = 2$
27	$3 = f^{-1}(27) \Rightarrow \log_3(27) =$
81	$4 = f^{-1}(81) \Rightarrow \log_3(81) = 4$

Every exponential function has an inverse. The inverses of exponential functions are called *logarithmic functions* (logarithms or logs for short).

> Definition: $\log_b x = y$ means $b^y = x$.
>
> The functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other.
> b is called the *base* of the logarithm.

Note: $\log_b x = y \Longleftrightarrow b^y = x$

Evaluating logarithms:

Example 2: Evaluate $\log_2 8$.

$\log_2 8 = \underline{3}$

$2^? = 8$

In other words, $\log_2 8$ is a number. What number is it?

answer: 3

This question is asking us to find a certain exponent. Specifically, "what exponent must I put on the 2 to give me 8?"

Said another way, "2 raised to what power is 8?"

Examples:

$\log_2 32 = \boxed{5}$

$2^? = 32$

$\log_5 1 = \boxed{0}$

$5^? = 1$ answer: 0

Recall:

$x^{-n} = \frac{1}{x^n}$

$x^{\frac{1}{n}} = \sqrt[n]{x}$

$\log_{10} 100 = \boxed{2}$

$10^? = 100$ answer: 2

$\log_3 \sqrt{3} = \boxed{\frac{1}{2}}$

$3^? = \sqrt{3} \Rightarrow 3^? = 3^{1/2}$

$\log_{\frac{1}{2}} 1 = \boxed{0}$

$\left(\frac{1}{2}\right)^? = 1$ answer:

$\log_3 \sqrt{3} = \boxed{\frac{1}{2}}$

$3^? = \sqrt{3} = 3^{\frac{1}{2}}$

$\log_5\left(\sqrt[4]{5}\right) = \boxed{\frac{1}{4}}$

$5^? = \sqrt[4]{5} = 5^{\frac{1}{4}}$

$\log_3\left(\frac{1}{9}\right) = \boxed{-2}$

$3^? = \frac{1}{9} = \frac{1}{3^2} = 3^{-2}$

$\log_{10}\left(\frac{1}{1000}\right) = \boxed{-3}$

$10^? = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$

$\log_2\left(\sqrt[3]{16}\right) = \boxed{\frac{4}{3}}$

$2^? = \sqrt[3]{16} = \sqrt[3]{2^4} = (2^4)^{1/3} = 2^{4 \cdot \frac{1}{3}}$

$= 2^{\frac{4}{3}}$

$\log_3\left(\frac{1}{\sqrt{3}}\right) = \boxed{-\frac{1}{2}}$

$3^? = \frac{1}{\sqrt{3}} = \frac{1}{3^{1/2}} = 3^{-\frac{1}{2}}$

$\log_{64} 8 = \boxed{\frac{1}{2}}$

$64^? = 8$

Note: $64 = 8^2$

$\sqrt{64} = 8$

$64^? = 8 = \sqrt{64} = 64^{\frac{1}{2}}$

Evaluating more complicated logarithms:

Example 3: $\log_4 32 =$ $\boxed{\frac{5}{2}}$

$4^{?} = 32$

32 is not a power of 4

use a variable: $4^{x} = 32$

write both sides as powers of same base: $(2^2)^x = 2^5$

$2^{2x} = 2^5$

Set exponents equal: $2x = 5$

$\frac{2x}{2} = \frac{5}{2}$

$x = \frac{5}{2}$

Example 4: $\log_9\left(\frac{1}{27}\right) =$ $\boxed{-\frac{3}{2}}$

$9^{?} = \frac{1}{27}$

$9^{x} = \frac{1}{27}$

$(3^2)^x = \frac{1}{3^3}$

$3^{2x} = 3^{-3}$

$2x = -3$ exponents equal

$\frac{2x}{2} = \frac{-3}{2}$

$x = -\frac{3}{2}$

The natural logarithm:

Remember we said *e* was a very important number? It is so important that the logarithmic function of base *e* has its own special notation and its own button on your calculator.

The logarithm of base *e* is called the *natural logarithm*, which is abbreviated "ln".

$$\boxed{\log_e x = \ln x}$$

$\log_e x = \ln x$

Example 5: $\ln e^4 =$ 4

This is asking $\log_e e^4 =$ 4

$e^{?} = e^4$ answer: 4

Example 6: $\ln\left(\frac{1}{e^3}\right) =$ -3

$e^{?} = \frac{1}{e^3} = e^{-3}$

both equivalent both are correct notation

$\ln\left(\frac{1}{e^3}\right) = \ln\left(e^{-3}\right) = \ln e^{-3} = \boxed{-3}$

Example 7: Evaluate $\ln\sqrt{e}$.

$$\ln\sqrt{e} = \ln e^{\frac{1}{2}} = \boxed{\frac{1}{2}}$$

Example 8: Simplify $\ln\left(\frac{1}{\sqrt[3]{e^5}}\right)$.

$$\ln\left(\frac{1}{\sqrt[3]{e^5}}\right) = \ln\left(\frac{1}{(e^5)^{\frac{1}{3}}}\right) = \ln\left(\frac{1}{e^{5/3}}\right) = \ln e^{-\frac{5}{3}} = \boxed{-\frac{5}{3}}$$

Example 9: Evaluate $\ln 1$.

$\ln 1 = \underline{\ ?\ }$

$e^{?} = 1$ answer: 0. So $\boxed{\ln 1 = 0}$

Example 10: Evaluate $\log_2(-4)$.

$2^{?} = -4$ ← This is impossible! No exponent on a 2 will give us a negative result.

So $\boxed{\log_2(-4) \text{ is not defined}}$

Example 11: Evaluate $\log_5 0$.

$5^{?} = 0$ ← This is impossible!

$\boxed{\log_5 0 \text{ is not defined}}$

IMPORTANT:

You cannot apply a logarithm to zero or to a negative number!!!

Ex: Find domain of $f(x) = \log_3(x-12)$.

$x - 12 > 0$

$x > 12$

$\boxed{\text{Domain: } (12, \infty)}$

Exponential and logarithmic forms for an equation:

Remember, $\log_b x = y$ means $b^y = x$.

$\log_b x = y \Longleftrightarrow b^y = x$

Logarithmic form: $\log_b x = y$
Exponential form: $b^y = x$

Example 12: Convert each of the following to exponential form.

a) $\log_{10} 1000 = 3$ $\boxed{10^3 = 1000}$

b) $\log_a 178 = w$ $\boxed{a^w = 178}$

c) $\log_7(y-3) = x$ $\boxed{7^x = y-3}$

d) $\log_x 6 = y^2 + 2$ $\boxed{x^{y^2+2} = 6}$

Example 13: Convert each of the following to logarithmic form.

a) $7^x = 23$ $7^x = 23 \Longleftrightarrow \boxed{\log_7 \underline{23} = \underline{x}}$

Check. $\log_7 23 = x$

$7^x = 23$

b) $y^{x-1} = 8$ $y^{x-1} = 8 \Longleftrightarrow \boxed{\log_y 8 = x-1}$

$\log_y 8 = x-1$

c) $x^8 = u$ $x^8 = u \Longleftrightarrow \boxed{\log_x u = 8}$

d) $(x-3)^2 = 6$

$\boxed{\log_{x-3} 6 = 2}$

More about the relationship between $f(x) = b^x$ and $g(x) = \log_b x$:

Because $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of one another, $f(g(x)) = x$ and $g(f(x)) = x$.

This gives us…

$$\boxed{\begin{array}{c} \log_b b^x = x \\ b^{\log_b x} = x \end{array}}$$

$$\log_b b^x = x$$
$$b^{\log_b x} = x$$

Example 14: Simplify $\log_3 3^{x+5}$.

$$\log_3 3^{x+5} = \boxed{x+5}$$

Example 15: Simplify $\log_2 2^{12}$.

$$\log_2 2^{12} = \boxed{12}$$

Example 16: Simplify $\ln e^{-2}$.

$$\ln e^{-2} = \log_e e^{-2} = \boxed{-2}$$

Example 17: Simplify $5^{\log_5 y}$.

$$5^{\log_5 y} = \boxed{y}$$

Example 18: Simplify $3^{\log_3 \sqrt{2}}$.

$$3^{\log_3 \sqrt{2}} = \boxed{\sqrt{2}}$$

Example 19: Simplify $e^{\ln(x^2+1)}$.

$$e^{\ln(x^2+1)} = e^{\log_e (x^2+1)} = \boxed{x^2+1}$$

The common logarithm:

Often $\log$ is used to mean $\log_{10}$. The logarithm of base 10 is called the *common logarithm*.

Example 20: Evaluate $\log \sqrt[3]{10}$.

$$\log \sqrt[3]{10} = \log_{10} \sqrt[3]{10} = \log_{10} 10^{\frac{1}{3}}$$
$$= \boxed{\frac{1}{3}}$$

Evaluating logs on your calculator:

Example 21: Evaluate $\log_{10} 72$ on your calculator.

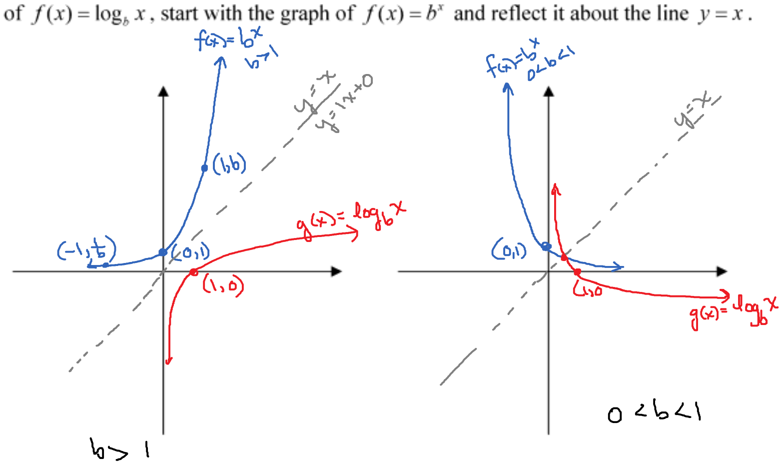
$$\log_{10} 72 = \log 72 \approx \boxed{1.85733}$$

Example 22: Evaluate $\ln 12$ on your calculator.

$$\boxed{\ln 12 \approx 2.48491}$$

Graphs of logarithmic functions:

To get the graph of $f(x) = \log_b x$, start with the graph of $f(x) = b^x$ and reflect it about the line $y = x$.
Why?



Facts about the graphs of $y = b^x$ and $y = \log_b x$:

$y = b^x$			$y = \log_b x$	
Domain:	$(-\infty, \infty)$	same	Domain:	$(0, \infty)$
Range:	$(0, \infty)$	same	Range:	$(-\infty, \infty)$
Asymptote:	the line $y=0$ (the x-axis)		Asymptote:	$x=0$ (the y-axis)
Passes through:	$(0,1)$, $(1,b)$, $(-1,\frac{1}{b})$		Passes through:	$(1,0)$, $(b,1)$, $(\frac{1}{b},-1)$

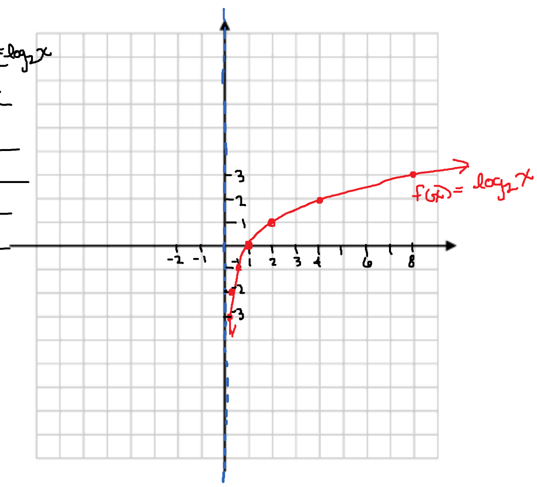
Example 23: Sketch the graph of $f(x) = \log_2 x$ by plotting points.

Start with $y = 2^x$

x	$y=2^x$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

for sure always do these 3

x	$f(x)=\log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



$\boxed{\text{Vertical asymptote: } x=0}$

Example 24: Graph $\cancel{f(x)=\log_3(x+3)}$.

Start with $f(x) = \log_3(x+3)$

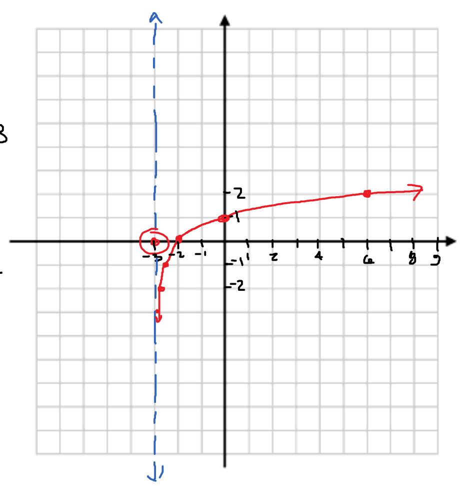
Parent function: $y = \log_3 x$, then shift it left 3.

Note: $x+3=0$
$x=-3 \Rightarrow$ shift left 3

Before we graph $y=\log_3 x$, let's find ordered pairs for $y=3^x$

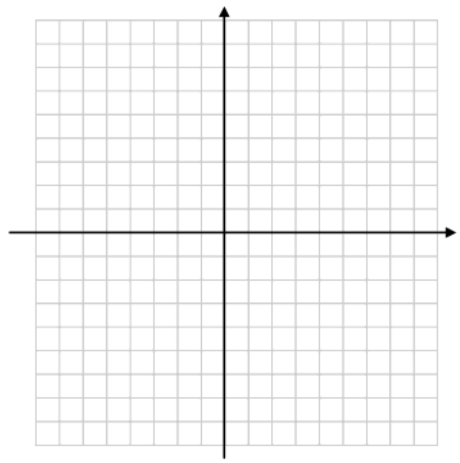
x	$y=3^x$
-2	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
-1	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$

x	$y=\log_3 x$
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

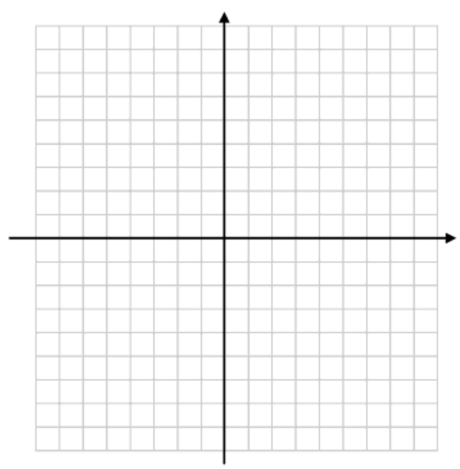


vertical asymptote: $\boxed{x=-3}$

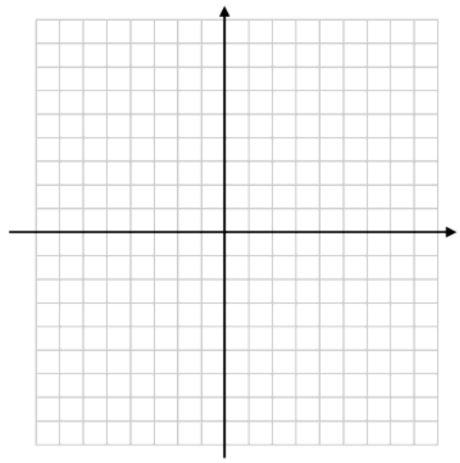
Example 25: Graph $y = -\ln x$.



Example 26: Graph $g(x) = \ln(x+2) - 1$.



Example 27: Graph $f(x) = -2 - \log_2(x-3)$.



Important: When graphing logarithmic and exponential functions, ALWAYS label the reference point with its coordinates. Also label the asymptote.

Example 28: Find the function of the form $y = \log_a x$ whose graph includes the point $(64, 3)$.

Finding the domain of logarithmic functions:

Example 29: Find the domain of $f(x) = \log_3(x-4)$.

Example 30: Find the domain of $f(x) = \log_5(x^2)$.

Example 31: Find the domain of $f(x) = \ln(x^2 + 6)$.

Example 32: Find the domain of $g(x) = \ln(3 - 2x)$.

Example 33: Find the domain of $h(x) = \ln(-x)$.

Solving simple logarithmic equations:

Example 34: Solve for x.

$$\log_2(x-1) = 5$$

Example 35: Solve for x.

$$\log_x 7 = \frac{1}{2}$$