1.4: Complex Numbers

Until now, we have not been able to solve equations where square roots of negative numbers occurred.

Ex: $x^2 + 1 = 0$
$\quad -1 \quad -1$
$x^2 = -1$

There are no real numbers that give you -1 when you square them.

$x^2 + 1 = 0$ has $\boxed{\text{No real solutions}}$

Ex: $x^2 - 9 = 0$
$\quad +9 \quad +9$
$x^2 = 9$

what value(s) of x make this equation true?

$x = 3 \Rightarrow (3)^2 = 9$
$9 = 9$

$x = -3 \Rightarrow (-3)^2 = 9$
$9 = 9$
true

$\boxed{\text{Sol'n Set: } \{3, -3\}}$

We introduce the imaginary numbers.

$$i = \sqrt{-1}$$

$$i^2 = -1$$

The imaginary numbers are multiples of i: $6i$, $-7i$, $14i$, $\frac{2}{3}i$, $i\sqrt{5}$

Putting together real numbers with imaginary numbers gives us the complex numbers.

All complex numbers have the form $a + bi$, where a and b are real numbers.

Examples of complex numbers: $2 + 5i$, $3 - 8i$, $1 + i$

For $a + bi$, a is called the real part
b is called the imaginary part.

Ex: $5 - 3i$ has real part 5 and imaginary part -3

To do arithmetic / algebra with complex numbers, we handle the i like an x, except we must remember that $i^2 = -1$.

Ex: Simplify.
$$(5+4i)+(2-i)$$
$$= 5+4i+2-i$$
$$= \boxed{7+3i}$$

Ex: Simplify.
$$2(-2+3i)-(1-5i)$$
$$= -4+6i-1+5i$$
$$= -5+11i$$

Ex: $(3-2i)(6+7i)$
$$= 18+21i-12i-14i^2$$
$$= 18+9i-14i^2$$
$$= 18+9i-14(-1) \longleftarrow i^2=-1$$
$$= 18+9i+14$$
$$= \boxed{32+9i}$$

Ex: Simplify.

a) $\sqrt{-7} = \sqrt{-1(7)} = \sqrt{-1}\sqrt{7} = \boxed{i\sqrt{7}}$

b) $\sqrt{-13} = \boxed{i\sqrt{13}}$ (could write $\sqrt{13}\,i$ but it is unclear)

c) $\sqrt{-4} = \sqrt{-1(4)} = \sqrt{-1}\sqrt{4} = i\cdot 2 = \boxed{2i}$

d) $\sqrt{-24} = i\sqrt{24} = i\sqrt{4\cdot 6} = i\sqrt{4}\sqrt{6} = i\cdot 2\sqrt{6} = \boxed{2i\sqrt{6}}$

Complex Conjugates

Definition: If $z = a + bi$, then the conjugate (or complex conjugate) of z is $\bar{z} = a - bi$.

Examples: The conjugate of $7 - 3i$ is $7 + 3i$
The conjugate of $-2 + i$ is $-2 - i$
The conjugate of $4i$ is $-4i$ (could write as $0 + 4i$ and $0 - 4i$)

If we multiply a complex number times its conjugate, the result is a real number.

Ex: $(5 + 2i)(5 - 2i)$

$= 25 - 10i + 10i - 4i^2$

$= 25 - 4i^2$

$= 25 - 4(-1)$

$= 25 + 4$

$= 29$

Note: $(5 + 2x)(5 - 2x)$
$= 25 - 4x^2$

Recall: Rationalize the denominator.

$$\frac{3}{5 - \sqrt{7}} = \frac{3}{5 - \sqrt{7}}\left(\frac{5 + \sqrt{7}}{5 + \sqrt{7}}\right)$$

Note: We're using $(A - B)(A + B) = A^2 - B^2$

$$= \frac{15 + 3\sqrt{7}}{25 + \cancel{5\sqrt{7}} - \cancel{5\sqrt{7}} - 7}$$

$$= \frac{15 + 3\sqrt{7}}{25 - 7} = \frac{15 + 3\sqrt{7}}{18}$$

$$= \frac{\cancel{3}^{1}(5 + \sqrt{7})}{\cancel{18}_{6}} = \boxed{\frac{5 + \sqrt{7}}{6}} = \boxed{\frac{5}{6} + \frac{\sqrt{7}}{6}}$$

Dividing complex numbers:

Ex: Write $\frac{5+4i}{2-3i}$ in the form $a+bi$.

$$\frac{5+4i}{2-3i} = \frac{5+4i}{2-3i}\left(\frac{2+3i}{2+3i}\right) = \frac{10+15i+8i+12i^2}{4+6i-6i-9i^2} = \frac{10+23i+12i^2}{4-9i^2}$$

$$= \frac{10+23i+12(-1)}{4-9(-1)} = \frac{10+23i-12}{4+9} = \frac{-2+23i}{13}$$

$$= \boxed{-\frac{2}{13}+\frac{23}{13}i}$$

Powers of i:

$i^2 = -1$

$i^3 = i^2 i = (-1)i = -i$

$i^4 = i^2 i^2 = (-1)(-1) = 1$

$i^5 = i^4 i = (1)i$ from $i^4=1$

$= i$

Because $i^4=1$, then whenever k is a multiple of 4, then $i^k=1$

Ex. $i^{12} = 1$

$i^{20} = 1$

$i^{44} = 1$

Ex: Simplify $i^{26} = i^{24} i^2 = 1 \cdot i^2 = i^2 = \boxed{-1}$

Ex: Simplify $i^{83} = i^{80} i^3 = 1\, i^3 = i^3 = i^2 i$

$= (-1)i = \boxed{-i}$