P.3.1

Radicals and Rational Exponents

A square root of a number a number that gives you the original number when you square it.

Ex: Find all square roots of 9.

$(3)^2 = 9$

$(-3)^2 = 9$

So 3 and -3 are square roots of 9.

The positive square root is called the principal square root and is written using $\sqrt{\;}$.

$$\boxed{\sqrt{9} = 3}$$

Ex: $\sqrt{36} = \boxed{6}$

Ex. What is the principal square root of 7?

$$\boxed{\sqrt{7}}$$

Note: $\sqrt{7} \approx 2.646$

Other roots:

$\sqrt[n]{a}$ is the n^{th} root of a.

$\sqrt[n]{\;\;}$ is the radical sign

The number or expression underneath the radical sign is called the radicand.

n is the index of the radical.

$\sqrt[3]{8} = 2$ because $(2)^3 = 8$

$\sqrt[4]{16} = 2$ because $(2)^4 = 16$

$\sqrt[3]{125} = 5$ because $(5)^3 = 125$

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Note: $(2)^4 = 16$

Also $(-2)^4 = 16$

If n is even, then $\sqrt[n]{\ }$ is the positive, or principal nth root.

$\sqrt[2]{81} = \sqrt{81} = 9$ (the square root)

because $(9)^2 = 81$

If n is odd, then $\sqrt[n]{a^n} = a$.

Ex: $\sqrt[3]{2^3} = \sqrt[3]{8} = \boxed{2}$

$(\sqrt[3]{8})^3 = 2^3 = \boxed{8}$

$(\sqrt[3]{7})^3 = \boxed{7}$

$\sqrt[3]{7^3} = \sqrt[3]{343} = \boxed{7}$

$\sqrt[3]{-8} = \boxed{-2}$ because $(-2)^3 = -8$

$\sqrt[3]{-64} = \boxed{-4}$ because $(-4)^3 = -64$

If n is even, then $\sqrt[n]{a^n} = |a|$

Ex: $\sqrt[2]{3^2} = \sqrt{3^2} = \sqrt{9} = \boxed{3}$

$\sqrt[2]{(-3)^2} = \sqrt{9} = \boxed{3}$

$\sqrt[4]{(-3)^4} = \sqrt[4]{81} = \boxed{3}$

$\sqrt[4]{(13)^4} = \boxed{13}$

$\sqrt[4]{(-13)^4} = \boxed{13}$

$\hookrightarrow \sqrt[4]{(-13)^4} = |-13| = \boxed{13}$

Recall:

$|5| = 5$

$|8| = 8$

$|-8| = 8$

$|-10| = 10$

$\sqrt{x^2} = |x|$

Try $x = -3$: $\sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = 3 \neq x$

Simplifying Radical Expressions

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ if n is odd.
if n is even, both a and b must be positive for this to work.

Also, two expressions with radicals are considered like terms if both the radicand and the index match.

Ex: $2\sqrt{5} + 7\sqrt{5} = \boxed{9\sqrt{5}}$

Ex: $2\sqrt{3} - 8\sqrt{6} + 4\sqrt{6} - 13\sqrt{3} + 6$

$= -11\sqrt{3} - 4\sqrt{6} + 6$

It is standard in most situations to "simplify" a square root by writing it with no perfect squares under the radical.

Ex: Simplify.
$\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\sqrt{6} = \boxed{2\sqrt{6}}$

Perfect Squares:
$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$

Ex: Simplify.
$\sqrt{18} + \sqrt{12} + \sqrt{32}$
$= \sqrt{9 \cdot 2} + \sqrt{4 \cdot 3} + \sqrt{16 \cdot 2}$
$= \sqrt{9}\sqrt{2} + \sqrt{4}\sqrt{3} + \sqrt{16}\sqrt{2}$
$= 3\sqrt{2} + 2\sqrt{3} + 4\sqrt{2}$

like terms

$= \boxed{7\sqrt{2} + 2\sqrt{3}}$

To simplify a cube root, we write it with no perfect cubes under the radical.

Perfect Cubes

$1^3 = 1$
$2^3 = 8$
$3^3 = 27$
$4^3 = 64$
$5^3 = 125$

Ex: Simplify.

$$\sqrt[3]{40} = \sqrt[3]{8 \cdot 5}$$
$$= \sqrt[3]{8}\,\sqrt[3]{5}$$
$$= \boxed{2\sqrt[3]{5}}$$

If we assume variables represent positive numbers, then we can also simplify square roots with variables.

Ex: Simplify. (in all these, assume variables represent positive numbers)

$$\sqrt{49x^2}$$
$$= \sqrt{49}\,\sqrt{x^2} = \boxed{7x}$$

Ex: $\sqrt{20x^6}$

$$= \sqrt{20}\,\sqrt{x^6}$$
$$= \sqrt{4 \cdot 5}\ x^3$$
$$= 2\sqrt{5}\,x^3 = \boxed{2x^3\sqrt{5}}$$

[because $(x^3)^2 = x^6$]

Ex: $\sqrt{40x^3}$

$$= \sqrt{40}\,\sqrt{x^3}$$
$$= \sqrt{4 \cdot 10}\,\sqrt{x^2 \cdot x}$$
$$= \sqrt{4}\sqrt{10}\,\sqrt{x^2}\sqrt{x}$$
$$= 2\sqrt{10}\ x\sqrt{x}$$
$$= 2x\sqrt{10}\sqrt{x} = \boxed{2x\sqrt{10x}}$$

Ex. $\sqrt{x^5 y^7 z^8}$

$$= \sqrt{x^4 x}\,\sqrt{y^6 y}\,\sqrt{z^8}$$
$$= \sqrt{x^4}\sqrt{x}\,\sqrt{y^6}\sqrt{y}\ \ z^4$$
$$= x^2\sqrt{x}\ \ y^3\sqrt{y}\ \ z^4 = x^2y^3z^4\sqrt{x}\sqrt{y}$$
$$= \boxed{x^2y^3z^4\sqrt{xy}}$$

Rationalizing Denominators

Rational Number: Can be written as a ratio (fraction) of 2 integers.

Examples of rational numbers: $\frac{2}{3}, \frac{7}{5}, -\frac{14}{3}, 5, -2, 0$

$5 = \frac{5}{1}$, $-2 = \frac{-2}{1}$, $0 = \frac{0}{1}$

$1.342, 0.73, 0.\bar{3}$

$1.342 = \frac{1342}{1000}$, $0.73 = \frac{73}{100}$, $0.\bar{3} = \frac{1}{3}$

Irrational Numbers: cannot be written as a ratio of integers.

Examples of irrational numbers: $\sqrt{3}, \sqrt{7}, \sqrt[3]{5}, \pi$

With radicals, it is conventional to write them so that the denominators are rational (denominator does not have radicals in it)

Examples: Simplify. Rationalize all denominators.

Ex: $\frac{2}{\sqrt{6}} = \frac{2}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{6}}{\sqrt{36}} = \frac{\cancel{2}^{1}\sqrt{6}}{\cancel{6}_{3}} = \boxed{\frac{\sqrt{6}}{3}}$

Note:

$\frac{1}{3} + \frac{2}{5}$

$= \frac{1}{3} \cdot \frac{5}{5} + \frac{2}{5} \cdot \frac{3}{3}$

$= \frac{5}{15} + \frac{6}{15}$

$= \boxed{\frac{11}{15}}$

Ex: $\frac{5}{\sqrt{20}} = \frac{5}{\sqrt{20}} \cdot \frac{\sqrt{20}}{\sqrt{20}} = \frac{\cancel{5}^{1}\sqrt{20}}{\cancel{20}_{4}}$

$= \frac{1\sqrt{20}}{4} = \frac{\sqrt{20}}{4} = \frac{\sqrt{4 \cdot 5}}{4} = \frac{\cancel{2}^{1}\sqrt{5}}{\cancel{4}_{2}} = \boxed{\frac{\sqrt{5}}{2}}$

OR $\frac{5}{\sqrt{20}} = \frac{5}{\sqrt{4 \cdot 5}} = \frac{5}{2\sqrt{5}} = \frac{5}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{5\sqrt{5}}{2 \cdot 5} = \frac{\cancel{5}^{1}\sqrt{5}}{\cancel{10}_{2}}$

$= \boxed{\frac{\sqrt{5}}{2}}$

Ex: $\frac{\sqrt{6}}{\sqrt{40}} = \frac{\sqrt{6}}{\sqrt{4 \cdot 10}} = \frac{\sqrt{6}}{2\sqrt{10}} = \frac{\sqrt{6}}{2\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{6}\sqrt{10}}{2\sqrt{10}\sqrt{10}}$

$= \frac{\sqrt{60}}{2 \cdot 10} = \frac{\sqrt{60}}{20} = \frac{\sqrt{4 \cdot 15}}{20} = \frac{\cancel{2}^{1}\sqrt{15}}{\cancel{20}_{10}}$

$= \boxed{\frac{\sqrt{15}}{10}}$

$$= \boxed{\frac{\sqrt{15}}{10}}$$

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Note: For positive numbers A and B,

$$\sqrt{AB} = \sqrt{A}\sqrt{B}$$

and $\sqrt{\frac{A}{B}} = \frac{\sqrt{A}}{\sqrt{B}}$

Ex: $\sqrt{\frac{4}{3}} = \frac{\sqrt{4}}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \boxed{\frac{2\sqrt{3}}{3}}$

Multiplication with Radicals

Ex: $\sqrt{7}(2-\sqrt{5}) = 2\sqrt{7} - \sqrt{7}\sqrt{5}$

$= \boxed{2\sqrt{7} - \sqrt{35}}$

Ex: $(\sqrt{3} - 4)(\sqrt{2} + 2\sqrt{5})$ multiply it out. "FOIL"

O I F L

$= \sqrt{3}\sqrt{2} + 2\sqrt{3}\sqrt{5} - 4\sqrt{2} - 8\sqrt{5}$

$= \boxed{\sqrt{6} + 2\sqrt{15} - 4\sqrt{2} - 8\sqrt{5}}$

Ex: $(3+\sqrt{7})(3-\sqrt{7})$

$= 9 - \cancel{3\sqrt{7}} + \cancel{3\sqrt{7}} - \sqrt{7}\sqrt{7}$

$= 9 - \sqrt{7}\sqrt{7}$

$= 9 - \sqrt{49} = 9 - 7 = \boxed{2}$

Note:
$(a+b)(a-b)$
$= a^2 - b^2$
(Difference of 2 squares)

Ex: Rationalize the denominator.

$$\frac{7}{6+\sqrt{3}}$$

We "multiply by the conjugate"

$3+\sqrt{7}$ and $3-\sqrt{7}$ are conjugates.

$6+\sqrt{3}$ and $6-\sqrt{3}$ are conjugate

$$= \frac{7}{6+\sqrt{3}} \cdot \frac{6-\sqrt{3}}{6-\sqrt{3}}$$

$$= \frac{7(6-\sqrt{3})}{(6+\sqrt{3})(6-\sqrt{3})}$$

$$= \frac{42-7\sqrt{3}}{36-\cancel{6\sqrt{3}}+\cancel{6\sqrt{3}}-\sqrt{3}\sqrt{3}}$$

$$= \frac{42-7\sqrt{3}}{36-3} = \boxed{\frac{42-7\sqrt{3}}{33}} = \boxed{\frac{7(6-\sqrt{3})}{33}}$$

both OK

Ex: Rationalize the denominator

$$\frac{1+\sqrt{3}}{\sqrt{5}-4} = \frac{1+\sqrt{3}}{\underbrace{\sqrt{5}-4}_{a-b}} \cdot \frac{\sqrt{5}+4}{\underbrace{\sqrt{5}+4}_{a+b}} = \frac{\sqrt{5}+4+\sqrt{15}+4\sqrt{3}}{5+\cancel{4\sqrt{5}}-\cancel{4\sqrt{5}}-16}$$

$$= \frac{\sqrt{5}+4+\sqrt{15}+4\sqrt{3}}{5-16} = \boxed{\frac{\sqrt{5}+4+\sqrt{15}+4\sqrt{3}}{-11}}$$

$$= \boxed{-\frac{\sqrt{5}+4+\sqrt{15}+4\sqrt{3}}{11}}$$

Rational Exponents (Fractional Exponents)

Rational number: can be written as a ratio (fraction) of 2 integers

$\boxed{\text{Definition: } a^{\frac{1}{n}} = \sqrt[n]{a}}$ (Definition of a fractional exponent)

Ex: $9^{\frac{1}{2}} = \sqrt[2]{9} = \sqrt{9} = \boxed{3}$

Recall: $\sqrt[3]{8} = \boxed{2}$ because $(2)^3 = 8$.

$\sqrt[3]{125} = \boxed{5}$ because $(5)^3 = 125$

$\sqrt[4]{81} = \boxed{3}$ because $(3)^4 = 81$

$\sqrt[2]{81} = \sqrt{81} = \boxed{9}$ because $(9)^2 = 81$

Using fractional Exponents, we can write these as: $8^{\frac{1}{3}} = 2$

$125^{\frac{1}{3}} = 5$

$81^{\frac{1}{4}} = 3$

$81^{\frac{1}{2}} = 9$

Ex: $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = \boxed{-2}$ because $(-2)^3 = -8$

Ex: $(-25)^{\frac{1}{2}} = \sqrt{-25}$ $\boxed{\text{is not a real number}}$ because $(\quad)^2 = -25$

Fact:
Even roots of negative numbers are not real numbers.
(we never get a negative when we square a real number).

Ex: $\sqrt{-64}$ is $\boxed{\text{not a real number}}$

Exponents written as $\frac{a}{b}$ can be split up:

Ex: Simplify. $8^{\frac{2}{3}} = ?$

$8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = (2)^2 = \boxed{4}$

Ex: Simplify $81^{3/4}$

$$81^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{81}\right)^3$$

$$= (3)^3$$

$$= \boxed{27}$$

Note:
$\sqrt[4]{81} = 3$
because $(3)^4 = 81$

Ex: $9^{\frac{5}{2}} = \left(9^{\frac{1}{2}}\right)^5 = \left(\sqrt{9}\right)^5 = (3)^5$

$$= \boxed{243}$$

$3^4 = 81$

$$\begin{array}{r} 81 \\ 3 \\ \hline 243 \end{array}$$