1.5: Quadratic Equations

A *quadratic* equation in x is one that can be written in the general form $ax^2 + bx + c = 0$, where $a \neq 0$.

quadratic term linear term constant term

Quadratic equations can be solved by:

- factoring.
- taking square roots.
- completing the square
- using the quadratic formula.

Solving a quadratic equation by factoring:

Solving a quadratic equation by factoring involves the *zero-product principle*.

Zero-Product Principle If the product of two algebraic expressions is zero, then at least one of the factors is equal to zero. If $AB = 0$, then $A = 0$ or $B = 0$. In other words, if multiplying two expressions results in 0, then at least one of them must be zero.

To solve an equation by factoring:

1. Move everything to one side (i.e. write in the form $ax^2 + bx + c = 0$). (write in standard form for a quadratic)
2. Factor the nonzero side.
3. Set each factor equal to zero.
4. Solve each of the resulting new equations.

Example 1: Solve $x^2 - 3x = 4$ by factoring.

$x^2 - 3x = 4$
$\quad -4 \quad -4$

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x - 4 = 0$ OR $x + 1 = 0$

$x = 4 \qquad x = -1$

Solution Set: $\boxed{\{4, -1\}}$

Example 2: Solve $3c^2 = 27c - 42$ by factoring.

$3c^2 = 27c - 42$
$-3c^2 \quad -3c^2$

$0 = -3c^2 + 27c - 42$

$0 = -3(c^2 - 9c + 14)$

$0 = -3(c-7)(c-2)$

$-3 = 0$ or $c - 7 = 0$ or $c - 2 = 0$

never true $\qquad c = 7 \qquad c = 2$

Sol'n Set: $\{7, 2\}$

also correct: $\boxed{\{2, 7\}}$

Remember:

* Expressions get simplified (or factored)
* Equations get solved.

Example 3: Solve $3A^2 - 10A = 8$.

Example 4: Solve $x^2 = 25x$.

$x^2 - 25x = 0$

$x(x - 25) = 0$

$x = 0$ or $x - 25 = 0$

$x = 25$

$\boxed{\{0, 25\}}$

Example 5: Solve $12t^2 + 5t - 2 = 0$.

Example 6: Solve $18t^2 + 25t + 4 = 0$.

The Square Root Property

If $u^2 = d$, then $u = \sqrt{d}$ or $u = -\sqrt{d}$.

Note: If $d > 0$, then both solutions are positive real numbers. If $d < 0$, then both solutions are non-real complex numbers.

Solving a quadratic equation by using the square root property:

1. Write it so that one side is a perfect square.
2. Take square roots of both sides, remembering the $\pm$. Both the positive and negative square roots make the equation true.

Example 7: Solve $x^2 = 100$ by factoring.

$x^2 - 100 = 0$
$(x+10)(x-10) = 0$
$x = -10$, $x = 10$

Sol'n Set. $\boxed{\{10, -10\}}$ or $\boxed{\{\pm 10\}}$ "plus or minus 10"

Example 8: Solve $x^2 = 100$ by taking square roots.

$x = \pm\sqrt{100}$
$x = \pm 10$

$\boxed{\{\pm 10\}}$ or $\boxed{\{10, -10\}}$

Note:
$\sqrt{a}$ is the **positive** number whose square is a. This means that $\sqrt{5}, \sqrt{6}, \sqrt{77},$ and $\sqrt{13}$ are all positive numbers.

Thus $-\sqrt{5}, -\sqrt{6}, -\sqrt{77},$ and $-\sqrt{13}$ are all negative numbers.

Question: What is $\sqrt{9}$?

3

Ex: $x^2 = -100$
No real solution
$x = \pm\sqrt{-100}$
$x = \pm 10i$
$\boxed{\{\pm 10i\}}$
Recall. $i = \sqrt{-1}$

Example 9: Solve $y^2 - 17 = 0$ by using the square root property.

$y^2 = 17$
$y = \pm\sqrt{17}$

Sol'n Set: $\boxed{\{-\sqrt{17}, \sqrt{17}\}}$ or $\boxed{\{\pm\sqrt{17}\}}$

Note.

$$(w+3)^2 = 25$$

$$\sqrt{(w+3)^2} = \sqrt{25}$$

this is not quite right. This has positive = positive

If you put a square root over both sides, you should also put $\pm$ on one of the sides.

$$\sqrt{(w+3)^2} = \pm\sqrt{25}$$

$$\pm\sqrt{w+3} = \sqrt{25}$$

are OK!

Example 10: Solve $(w+3)^2 = 25$ by using the square root property.

$$\sqrt{(w+3)^2} = \pm\sqrt{25}$$

$$w+3 = \pm\sqrt{25}$$

$$w+3 = \pm 5$$

$$-3 \qquad -3$$

$$w = -3 \pm 5$$

$w = -3+5$ or $w = -3-5$

$w = 2$ or $w = -8$

Sol'n Set: $\boxed{\{2, -8\}}$

Example 11: Solve $x^2 + 64 = 0$ by using the square root property.

$$x^2 = -64$$

$$\sqrt{x^2} = \pm\sqrt{-64}$$

$$x = \pm\sqrt{64}\sqrt{-1}$$

$$x = \pm 8i$$

Sol'n Set: $\boxed{\{\pm 8i\}}$

Example 12: Solve $x^2 + 45 = 0$ by using the square root property.

$$x^2 = -45$$

$$x = \pm\sqrt{-45}$$

$$= \pm\sqrt{45}\, i$$

$$= \pm i\sqrt{45}$$

$$x = \pm i\sqrt{9 \cdot 5}$$

$$x = \pm i\sqrt{9}\sqrt{5}$$

$$x = \pm 3i\sqrt{5}$$

$\boxed{\{\pm 3i\sqrt{5}\}}$

Solving a quadratic equation by completing the square:

To "complete the square" on a quadratic equation:

1. Rewrite it with the constant term on the right side.
2. Divide both sides by the coefficient of x^2. You should now have something in the form $x^2 + bx = c$.
3. Add $\left(\frac{b}{2}\right)^2$ to both sides. This will make the left side into a perfect square.
4. Factor and solve by taking square roots.

Example 13: Solve $x^2 + 8x - 10 = 0$ by completing the square.

$$x^2 + 8x = 10$$

$$x^2 + 8x + \_\_\_ = 10 + \_\_\_$$

$$x^2 + 8x + 16 = 10 + 16$$

$$(x+4)^2 = 26$$

$$\sqrt{(x+4)^2} = \pm\sqrt{26}$$

$$x + 4 = \pm\sqrt{26}$$

$$x = -4 \pm\sqrt{26}$$

Sol'n Set: $\boxed{\{-4 \pm \sqrt{26}\}}$

$x \approx -1.099, -9.099$

Scratch:

$$\left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

Factor $x^2 + 8x + 16$

$(x+4)(x+4)$

$(x+4)^2$

16

2·8

1·16

4·4

Check:

$(x+4)(x+4)$

$x^2 + 4x + 4x + 16$

$x^2 + 8x + 16$ ✓

Note: $\sqrt{\frac{A}{B}} = \frac{\sqrt{A}}{\sqrt{B}}$ as long as $A, B > 0$

$\sqrt{AB} = \sqrt{A}\sqrt{B}$ as long as $A, B > 0$

Example 14: Solve $3x^2 - 18x + 1 = 0$ by completing the square.

$3x^2 - 18x = -1$

(Divide both sides by 3)

$\frac{3x^2}{3} - \frac{18x}{3} = \frac{-1}{3}$

$x^2 - 6x = -\frac{1}{3}$

$x^2 - 6x + \_\_\_ = -\frac{1}{3} + \_\_\_$

$x^2 - 6x + 9 = -\frac{1}{3} + 9$

Scratch: $\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$

$(x - 3)^2 = -\frac{1}{3} + \frac{9}{1}\left(\frac{3}{3}\right)$

$(x-3)^2 = -\frac{1}{3} + \frac{27}{3}$

$(x-3)^2 = \frac{26}{3}$

$x - 3 = \pm\sqrt{\frac{26}{3}}$

$x = 3 \pm \frac{\sqrt{26}}{\sqrt{3}}$

$x = 3 \pm \frac{\sqrt{26}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$

$x = 3 \pm \frac{\sqrt{78}}{3}$

$x = \frac{3}{1}\left(\frac{3}{3}\right) \pm \frac{\sqrt{78}}{3}$

$x = \frac{9}{3} \pm \frac{\sqrt{78}}{3} = \frac{9 \pm \sqrt{78}}{3}$

$\boxed{\left\{\frac{9 \pm \sqrt{78}}{3}\right\}}$

Example 15: Solve $x^2 - 7x - 5 = 0$ by completing the square.

Scratch: $\left(\frac{-7}{2}\right)^2 = \frac{49}{4}$

$x^2 - 7x - 5 = 0$

$x^2 - 7x = 5$

$x^2 - 7x + \_\_\_ = 5 + \_\_\_$

$x^2 - 7x + \frac{49}{4} = 5 + \frac{49}{4}$

$\left(x - \frac{7}{2}\right)^2 = \frac{5 \cdot 4}{1 \cdot 4} + \frac{49}{4}$

$\left(x - \frac{7}{2}\right)^2 = \frac{20}{4} + \frac{49}{4}$

$\left(x - \frac{7}{2}\right)^2 = \frac{69}{4}$

$x - \frac{7}{2} = \pm\sqrt{\frac{69}{4}}$

$x - \frac{7}{2} = \pm\frac{\sqrt{69}}{2}$

$x = \frac{7}{2} \pm \frac{\sqrt{69}}{2}$

$x = \frac{7 \pm \sqrt{69}}{2}$

$\boxed{\left\{\frac{7 \pm \sqrt{69}}{2}\right\}}$

Solving a quadratic equation using the quadratic formula:

Memorize this!

> The solutions to the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are given by
>
> $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This is called the <u>quadratic formula</u>.

Example 16: Solve $x^2 - 3x = 4$ using the quadratic formula.

$a = 1$
$b = -3$
$c = -4$

$x^2 - 3x - 4 = 0$

$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$

$x = \frac{3 \pm \sqrt{9 + 16}}{2}$

$x = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2}$

$x = \frac{3+5}{2}, \frac{3-5}{2}$

$x = \frac{8}{2}, \frac{-2}{2} \implies x = 4, -1$

Sol'n Set: $\boxed{\{4, -1\}}$

Rational solutions $\Rightarrow$ it would have factored

Example 17: Solve $t^2+3=5t$ using the quadratic formula.

$t^2-5t+3=0$

$a=1$
$b=-5$
$c=3$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$t=\frac{5\pm\sqrt{(-5)^2-4(1)(3)}}{2(1)}$

$t=\frac{5\pm\sqrt{25-12}}{2}=\frac{5\pm\sqrt{13}}{2}$

Sol'n Set: $\boxed{\left\{\frac{5\pm\sqrt{13}}{2}\right\}}$

Example 18: Solve $3x^2+2x=-2$.

Example 19: Solve $u^2-6u+9=0$.

Example 20: Solve $3x^2-4x=-1$.

Example 21: Solve $4y^2 - 3y = 10$.

Using the discriminant to determine the number of real solutions:

The *discriminant* is the expression under the square root sign in the quadratic formula.
(i.e. it=s the $b^2 - 4ac$.)

- If $b^2 - 4ac < 0$, the equation has no real solutions. There are two complex solutions, which are complex conjugates of one another.
- If $b^2 - 4ac > 0$, the equation has two real solutions.
- If $b^2 - 4ac = 0$, the equation has exactly one real solution.
- If $b^2 - 4ac$ is a perfect square (positive), and all the coefficients are rational, the equation has two rational solutions. (solutions are fractions or integersBno square roots)

Example 22: Describe the solutions of $4x^2 - 3x + 10 = 0$. Solve it.

$a = 4$
$b = -3$
$c = 10$

$b^2 - 4ac = (-3)^2 - 4(4)(10)$
$= 9 - 160 = -151$. This is the part under the square root sign.

So $\boxed{\text{no real solutions.}}$ There are 2 complex solutions.

$$x = \frac{3 \pm \sqrt{-151}}{2(4)} = \frac{3 \pm i\sqrt{151}}{8}$$

Sol'n Set:

$$\boxed{\left\{\frac{3 \pm i\sqrt{151}}{8}\right\}}$$

Example 23: Describe the solutions of $5c^2 - 10c + 1 = 0$. Solve it.

$a = 5$
$b = -10$
$c = 1$

$b^2 - 4ac = (-10)^2 - 4(5)(1)$
$= 100 - 20 = 80$

$\boxed{\text{Two real non-rational solutions}}$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{80}}{2(5)} = \frac{10 \pm \sqrt{4 \cdot 20}}{10} = \frac{10 \pm 2\sqrt{20}}{10}$$

$$= \frac{10 \pm 2\sqrt{4 \cdot 5}}{10} = \frac{10 \pm 2 \cdot 2\sqrt{5}}{10} = \frac{10 \pm 4\sqrt{5}}{10}$$

$$= \frac{\cancel{2}^{1}(5 \pm 2\sqrt{5})}{\cancel{10}_{5}} = \frac{5 \pm 2\sqrt{5}}{5}$$

Sol'n Set: $\boxed{\left\{\frac{5 \pm 2\sqrt{5}}{5}\right\}}$

Example 24: Describe the solutions of $6x^2 + x = 12$. Solve it.

$6x^2 + x - 12 = 0$

$a = 6$
$b = 1$
$c = -12$

$b^2 - 4ac = (1)^2 - 4(6)(-12)$
$= 1 + 24(12)$
$= 1 + 288 = 289 \Rightarrow$ $\boxed{\text{2 real solutions; they are rational}}$

$289 = 17^2$, so a perfect square

$$x = \frac{-1 \pm \sqrt{289}}{2(6)} = \frac{-1 \pm 17}{12}$$

1st solution: $x = \frac{-1 + 17}{12} = \frac{16}{12} = \frac{4}{3}$

2nd solution: $x = \frac{-1 - 17}{12} = -\frac{18}{12} = -\frac{3}{2}$

Sol'n Set: $\boxed{\left\{-\frac{3}{2}, \frac{4}{3}\right\}}$

It would have factored!

Example 25: Describe the solutions of $25a^2 - 30a + 9 = 0$. Solve it.

$b^2 - 4ac = (-30)^2 - 4(25)(9)$
$= 900 - 900$
$= 0 \Rightarrow$ $\boxed{\text{1 real rational solutions}}$