1.5: Quadratic Equations

A *quadratic* equation in *x* is one that can be written in the general form $ax^2 + bx + c = 0$, where $a \neq 0$.

Quadratic equations can be solved by:

- factoring.
- taking square roots.
- completing the square
- using the quadratic formula.

Solving a quadratic equation by factoring:

Solving a quadratic equation by factoring involves the *zero-product principle*.

> Zero-Product Principle
> If the product of two algebraic expressions is zero, then at least one of the factors is equal to zero.
> If $AB = 0$, then $A = 0$ or $B = 0$.
>
> In other words, if multiplying two expressions results in 0, then at least one of them must be zero.

To solve an equation by factoring:

1. Move everything to one side (i.e. write in the form $ax^2 + bx + c = 0$).
2. Factor the nonzero side.
3. Set each factor equal to zero.
4. Solve each of the resulting new equations.

Example 1: Solve $x^2 - 3x = 4$ by factoring.

Example 2: Solve $3c^2 = 27c - 42$ by factoring.

Example 3: Solve $3A^2 - 10A = 8$.

Example 4: Solve $x^2 = 25x$.

Example 5: Solve $12t^2 + 5t - 2 = 0$.

Example 6: Solve $18t^2 + 25t + 4 = 0$.

The Square Root Property

If $u^2 = d$, then $u = \sqrt{d}$ or $u = -\sqrt{d}$.

Note: If $d > 0$, then both solutions are positive real numbers. If $d < 0$, then both solutions are non-real complex numbers.

Solving a quadratic equation by using the square root property:

1. Write it so that one side is a perfect square.

2. Take square roots of both sides, remembering the $\pm$. Both the positive and negative square roots make the equation true.

Example 7: Solve $x^2 = 100$ by factoring.

Example 8: Solve $x^2 = 100$ by taking square roots.

Note:
$\sqrt{a}$ is the **positive** number whose square is a. This means that $\sqrt{5}, \sqrt{6}, \sqrt{77},$ and $\sqrt{13}$ are all positive numbers.

Thus $-\sqrt{5}, -\sqrt{6}, -\sqrt{77},$ and $-\sqrt{13}$ are all negative numbers.

Question: What is $\sqrt{9}$?

Example 9: Solve $y^2 - 17 = 0$ by using the square root property.

Example 10: Solve $(w+3)^2 = 25$ by using the square root property.

Example 11: Solve $x^2 + 64 = 0$ by using the square root property.

Example 12: Solve $x^2 + 45 = 0$ by using the square root property.

Solving a quadratic equation by completing the square:

To "complete the square on a quadratic equation:

1. Rewrite it with the constant term on the right side.
2. Divide both sides by the coefficient of x^2. You should now have something in the form $x^2 + bx = c$.
3. Add $\frac{b}{2}$ to both sides. This will make the left side into a perfect square.
4. Factor and solve by taking square roots.

Example 13: Solve $x^2 + 8x - 10 = 0$ by completing the square.

Example 14: Solve $3x^2 - 18x + 1 = 0$ by completing the square.

Example 15: Solve $x^2 - 7x - 5 = 0$ by completing the square.

Solving a quadratic equation using the quadratic formula:

The solutions to the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$. This is called the quadratic formula.

Example 16: Solve $x^2 - 3x = 4$ using the quadratic formula.

Example 17: Solve $t^2 + 3 = 5t$ using the quadratic formula.

Example 18: Solve $3x^2 + 2x = -2$.

Example 19: Solve $u^2 - 6u + 9 = 0$.

Example 20: Solve $3x^2 - 4x = -1$.

Example 21: Solve $4y^2 - 3y = 10$.

Using the discriminant to determine the number of real solutions:

The *discriminant* is the expression under the square root sign in the quadratic formula. (i.e. it=s the $b^2 - 4ac$.)

- If $b^2 - 4ac < 0$, the equation has no real solutions. There are two complex solutions, which are complex conjugates of one another.
- If $b^2 - 4ac > 0$, the equation has two real solutions.
- If $b^2 - 4ac = 0$, the equation has exactly one real solution.
- If $b^2 - 4ac$ is a perfect square (positive), and all the coefficients are rational, the equation has two rational solutions. (solutions are fractions or integersBno square roots)

Example 22: Describe the solutions of $4x^2 - 3x + 10 = 0$. Solve it.

Example 23: Describe the solutions of $5c^2 - 10c + 1 = 0$. Solve it.

Example 24: Describe the solutions of $6x^2 + x = 12$. Solve it.

Example 25: Describe the solutions of $25a^2 - 30a + 9 = 0$. Solve it.