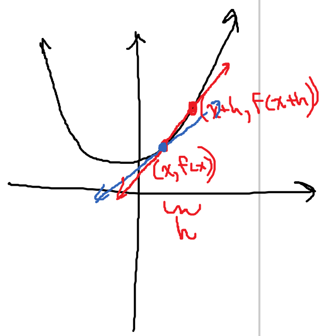
2.2: More on Functions and Their Graphs

The difference quotient:

The expression $\frac{f(x+h)-f(x)}{h}$ is called a *difference quotient* for *f*. The difference quotient is important in our understanding of the rate of change of a function.

$$\frac{f(x+h)-f(x)}{h} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \text{slope}$$

Example 1: For the function $f(x)=7-3x$, find and simplify the difference quotient.



$$\frac{f(x+h)-f(x)}{h} = \frac{\overbrace{7-3\boxed{x+h}}^{f(x+h)} - \overbrace{(7-3\boxed{x})}^{f(x)}}{h}$$

$$= \frac{7-3(x+h)-(7-3x)}{h} = \frac{\cancel{7}-\cancel{3x}-3h-\cancel{7}+\cancel{3x}}{h}$$

$$= \frac{-3\cancel{h}}{\cancel{h}} = \boxed{-3}$$

Example 2: For the function $f(x)=x^2+2x-1$, find and simplify the difference quotient.

$$\frac{f(x+h)-f(x)}{h} = \frac{\overbrace{\boxed{x+h}^2+2\boxed{x+h}-1}^{f(x+h)} - \overbrace{(\boxed{x}^2+2\boxed{x}-1)}^{f(x)}}{h}$$

$$= \frac{\overbrace{(x+h)^2+2(x+h)-1}^{f(x+h)} - \overbrace{(x^2+2x-1)}^{f(x)}}{h} = \frac{\cancel{x^2}+2xh+h^2+\cancel{2x}+2h-\cancel{1}-\cancel{x^2}-\cancel{2x}+\cancel{1}}{h}$$

$$= \frac{2xh+h^2+2h}{h} = \frac{\cancel{h}(2x+h+2)}{\cancel{h}} = \boxed{2x+h+2}$$

Piecewise Defined Functions:

(Functions defined in "pieces")

Example 3: $f(x)=\begin{cases}3-x, & \text{if } x\le 1\\ x^2, & \text{if } x>1\end{cases}$

There are two different rules. The rule we use depends on which x we put in.

Calculate $f(4)$, $f(-5)$, $f(1)$.

$x=4$ is in 2^{nd} category $(x>1)$, so use $y=x^2$

$x=-5$ is in 1^{st} category $(x\le 1)$, so use $y=3-x$

$x=1$ is in 1^{st} category $(x\le 1)$, so use $y=3-x$

$f(4)=(4)^2=\boxed{16}$

$f(-5)=3-(-5)=3+5=\boxed{8}$

$f(1)=3-1=\boxed{2}$

Example 4: Write $f(x)=|x|$ as a piecewise function.

$$f(x)=|x|=\begin{cases}x & \text{if } x\ge 0\\ -x & \text{if } x<0\end{cases}$$

Example 5: $f(x)=\begin{cases}3x+4, & \text{if } x<-2\\ 7 & \text{if } x=-2\\ x^2+1, & \text{if } x>-2\end{cases}$

Calculate $f(3)$, $f(-3)$, $f(-2)$.

$f(3)=x^2+1\Big|_{x=3}=(3)^2+1=\boxed{10}$

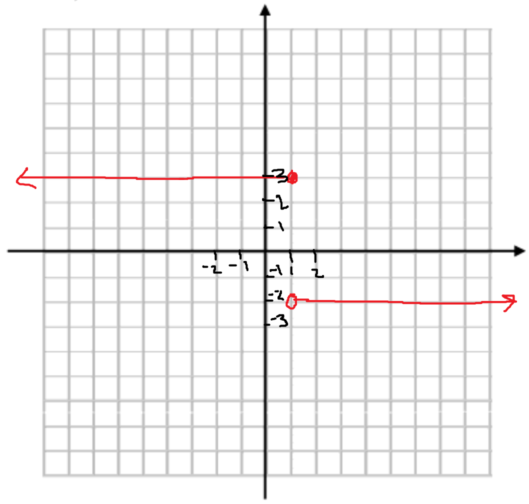
$f(-3)=3x+4\Big|_{x=-3}=3(-3)+4=-9+4=\boxed{-5}$

$f(-2)=\boxed{7}$

2.2.3

Graphing piecewise-defined functions: (Piece functions)

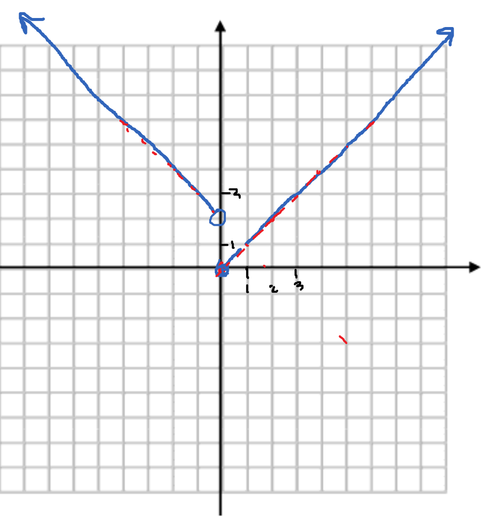
Example 1: $f(x) = \begin{cases} 3 & \text{if } x \le 1 \\ -2 & \text{if } x > 1 \end{cases}$



left of $x = 1$: $y = 3$

right of $x = 1$: $y = -2$

Example 2: $f(x) = \begin{cases} x & \text{if } x \ge 0 \\ -x + 2 & \text{if } x < 0 \end{cases}$



right of $x = 0$: $y = x$

$y = 1x + 0 \Rightarrow$ slope: $m = 1 = \frac{+1}{1}$ ↗

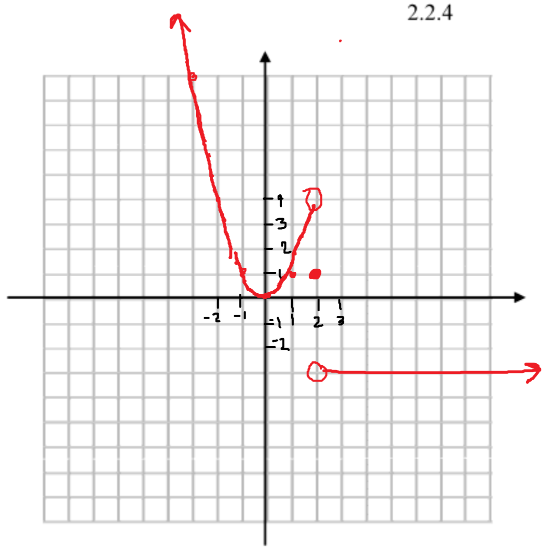
y-int: $b = 0$

left of $x = 0$: $y = -x + 2$

$y = -1x + 2 \Rightarrow$ slope: $m = -\frac{1}{1}$ ↘

y-int: $b = 2$

Example 3: $g(x)=\begin{cases}-3 & \text{if } x>2\\ x^2 & \text{if } x<2\\ 1 & \text{if } x=2\end{cases}$



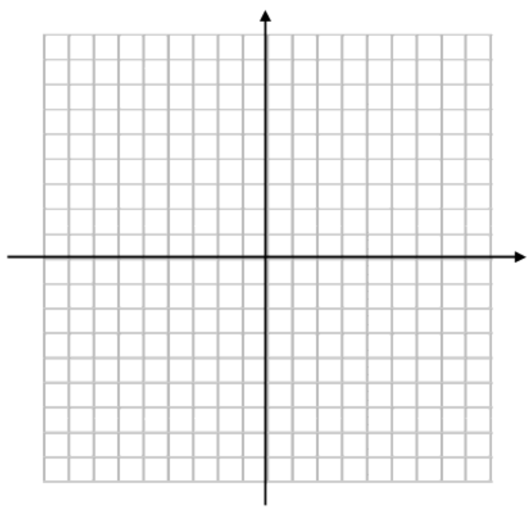
right of 2: $y=-3$
constant function

left of 2: $y=x^2$

x	$y=x^2$
0	$(0)^2=0$
1	$(1)^2=1$
2	$(2)^2=4$
3	$(3)^2=9$
-1	$(-1)^2=1$
-2	$(-2)^2=4$
-3	$(-3)^2=9$

exactly at 2: $y=1$

Example 4: $f(x)=\begin{cases}x^3 & \text{if } x\ge 0\\ 3x+4 & \text{if } x<0\end{cases}$



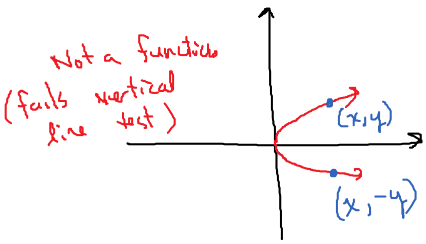
Symmetry:

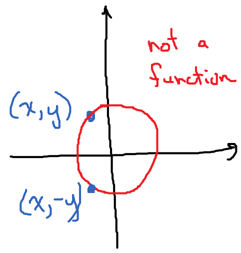
A graph is *symmetric with respect to the x-axis* if, for every point (x, y) on the graph, the point $(x, -y)$ is also on the graph.

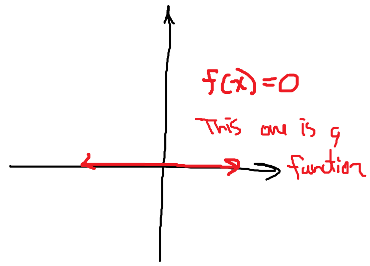
A graph is *symmetric with respect to the y-axis* if, for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

A graph is *symmetric with respect to the origin* if, for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.

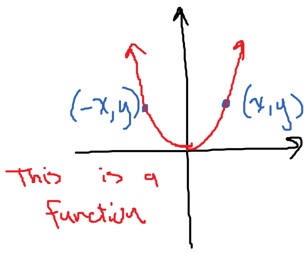
<u>Symmetry about the x-axis:</u>

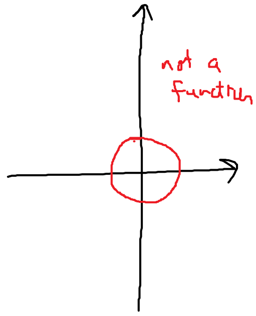


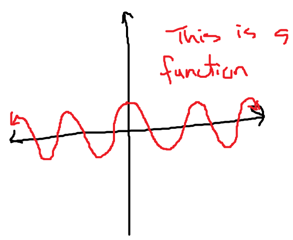




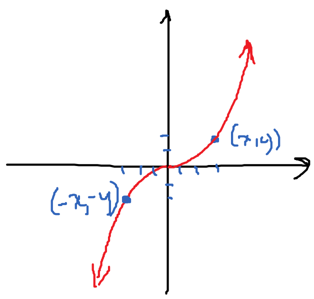
<u>Symmetry about the y-axis:</u>

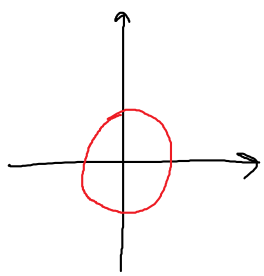


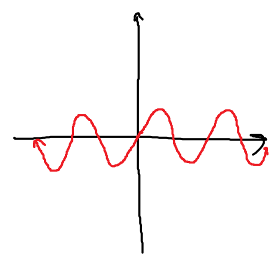




<u>Symmetry about the origin:</u>







Example 6: Is this equation symmetric around (a) the x-axis? (b) the y-axis? (c) the origin?

$$y = x^2 + 8$$

(a) x-axis symmetry:
replace y with $-y$: $-y = x^2 + 8$
Cannot be rearranged to get original $y = x^2 + 8$.
So does not have x-axis symmetry

(b) y-axis symmetry:
replace x by $-x$:
$y = (-x)^2 + 8$
$y = x^2 + 8$
Same as original, so it has y-axis symmetry

(c) Symmetry around origin:
replace x by $-x$ and y by $-y$.
$-y = (-x)^2 + 8$
$-y = x^2 + 8$ not equal to the original. So does not have symmetry about the origin

Example 7: Is this equation symmetric around (a) the x-axis? (b) the y-axis? (c) the origin?

$$x = y^2 + 8$$

(a) x-axis symmetry:
replace y with $-y$:
$x = (-y)^2 + 8$
$x = y^2 + 8$ same as original
Has x-axis symmetry

(b) y-axis symmetry:
replace x with $-x$
$-x = y^2 + 8$
not equal to original
Not symmetric about y-axis

(c) symmetry about origin:
replace x with $-x$ and y with $-y$:
$-x = (-y)^2 + 8$
$-x = y^2 + 8$ not equal to original
Not symmetric about the origin

Example 8: Is this equation symmetric around (a) the x-axis? (b) the y-axis? (c) the origin?

$$x^2 = y^2 + 8$$

Symmetric about the x-axis, y-axis, and origin.

Even and odd functions:

A function f is an <u>even function</u> if $f(-x) = f(x)$ for every x in the domain of f. The graph of an even function is symmetric with respect to the y-axis.

A function f is an <u>odd function</u> if $f(-x) = -f(x)$ for every x in the domain of f. The graph of an even function is symmetric with respect to the origin.

Example 9: Is the function $f(x) = 2 + x^2 + 3x^4$ even, odd, or neither?

Is it an even function? Even functions have y-axis symmetry, so replace x with $-x$:

$f(-x) = 2 + (-x)^2 + 3(-x)^4$

$f(-x) = 2 + x^2 + 3x^4$

$f(-x) = f(x)$, so it has y-axis symmetry, so $\boxed{\text{an even function}}$

Is it an odd function? Odd functions have symmetry about the origin, so replace x with $-x$ and see if y ~~with~~ is $-y$. want to see if $f(-x) = -f(x)$.

$f(-x) = 2 + (-x)^2 + 3(-x)^4$

$= 2 + x^2 + 3x^4$

$-f(x) = -(2 + x^2 + 3x^4)$

$= -2 - x^2 - 3x^4 \neq f(-x)$

$\boxed{\text{Does not have origin symmetry.}}$

Example 10: Is the function $f(x) = x^3 + x^5$ even, odd, or neither?

Is it an even function? replace x with $-x$:

$f(-x) = (-x)^3 + (-x)^5$

$= -x^3 - x^5$

$\neq f(x)$

$\boxed{\text{Not an even function}}$

Is it an odd function? Compare $f(-x)$ with $-f(x)$

$f(-x) = (-x)^3 + (-x)^5$

$= -x^3 - x^5$

$-f(x) = -(x^3 + x^5)$

$= -x^3 - x^5$

these are equal!

$f(-x) = -f(x)$, so therefore f has symmetry around origin, $\boxed{f \text{ is an odd function}}$

Example 11: Is the function $f(x) = x^3 + 1$ even, odd, or neither?

Is it an even function? Replace x with $-x$:

$f(-x) = (-x)^3 + 1$

$= -x^3 + 1$

$\neq f(x)$, so $\boxed{\text{not an even function}}$

Is it an odd function? Compare $f(-x)$ with $-f(x)$

$f(-x) = (-x)^3 + 1$

$= -x^3 + 1$

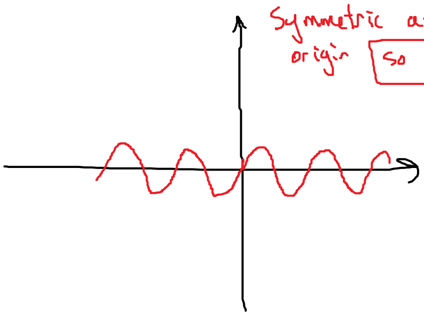
$-f(x) = -(x^3 + 1)$

$= -x^3 - 1$

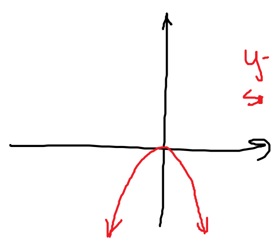
not equal!, so $\boxed{\text{not an odd function}}$

$\boxed{\text{This function is neither odd nor even}}$

Example 12: State whether each is the graph of a function that is odd, even or neither.



Symmetric around origin $\boxed{\text{so an odd function}}$



y-axis symmetry, so it is an $\boxed{\text{even function}}$