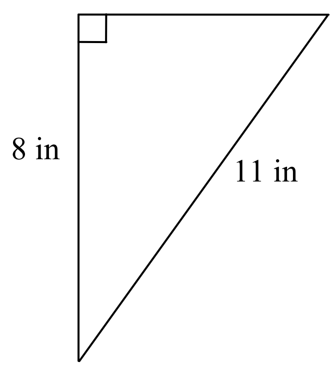
2.8: Distance and Midpoint Formulas; Circles

The Pythagorean Theorem

If a right triangle has hypotenuse of length c, and legs of length a and b, then $a^2 + b^2 = c^2$.

The converse of the Pythagorean Theorem is also true, though less well known: If the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third side, then the triangle is a right triangle.

Example 1: Find the length of the missing side.



Finding the distance between two points:

Example 2: Find the distance between $(1,3)$ and $(4,9)$.

The *distance formula* comes from the Pythagorean Theorem.

The Distance Formula

The distance between the two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

Example 3: Find the distance between $(-3,1)$ and $(-6,-3)$.

Example 4: Find the distance between $\left(\frac{1}{2}, 2\right)$ and $\left(-\frac{5}{2}, \frac{2}{3}\right)$.

Midpoints of line segments:

Definition: The *midpoint* of a line segment is the point on the segment that is equally distant from each end.

To find the midpoint:

To find the *x*-coordinate of the midpoint, just average the two *x*-coordinates.
To find the *y*-coordinate of the midpoint, just average the two *y*-coordinates.

Example 5: Find the midpoint of the line segment joining $\left(\frac{2}{5}, -4\right)$ and (4 ,7).

Midpoint Formula: The midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 6: If $M(-2, 8)$ is the midpoint of the line segment joining points *A* and *B*, and *A* has coordinates $(7, -3)$, find the coordinates of *B*.

Circles:

Definition: A *circle* is the set of all points that are equally distant from a fixed point. The fixed point is the *center*. The distance from the center to a point on the circle is called the *radius*.

Recall: The distance formula is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

If we let (h, k) represent the center and (x, y) represent a point on the circle, then the radius r is the distance between them.

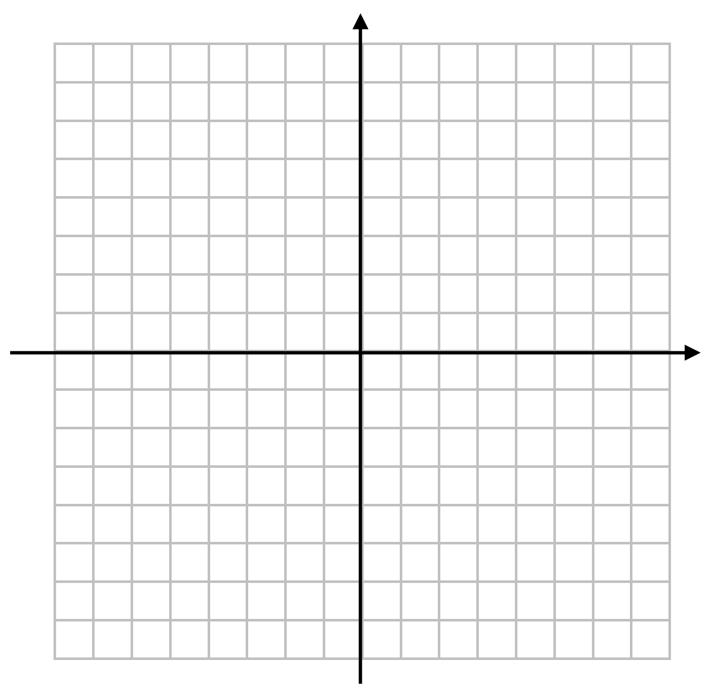
So,
$$r = \sqrt{(x-h)^2 + (y-k)^2}$$
$$r^2 = (x-h)^2 + (y-k)^2$$

Standard form for the equation of a circle:

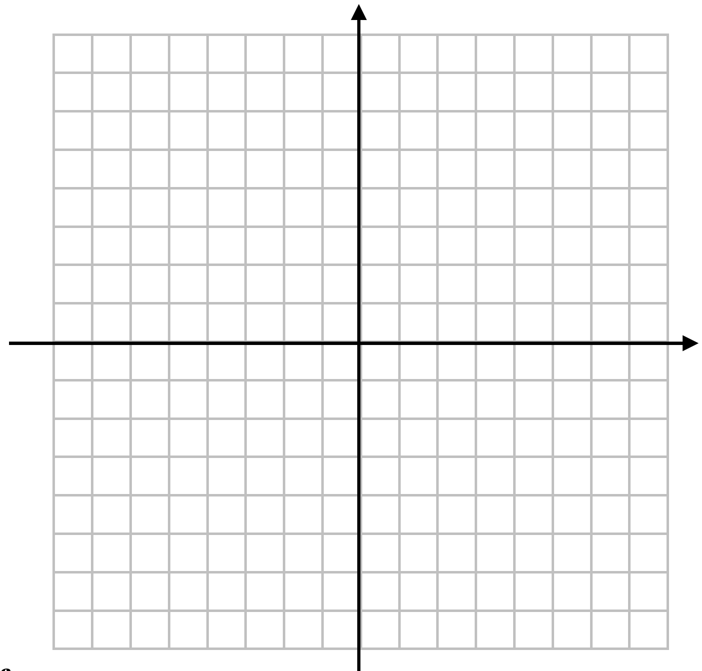
$(x-h)^2 + (y-k)^2 = r^2$, where

r is the radius and (h, k) is the center.

Example 1: Write the equation for a circle of radius 4 centered at $(2, -3)$. Graph it.



Example 2: Find the center and radius of $(x+3)^2+(y-1)^2=25$. Graph it.

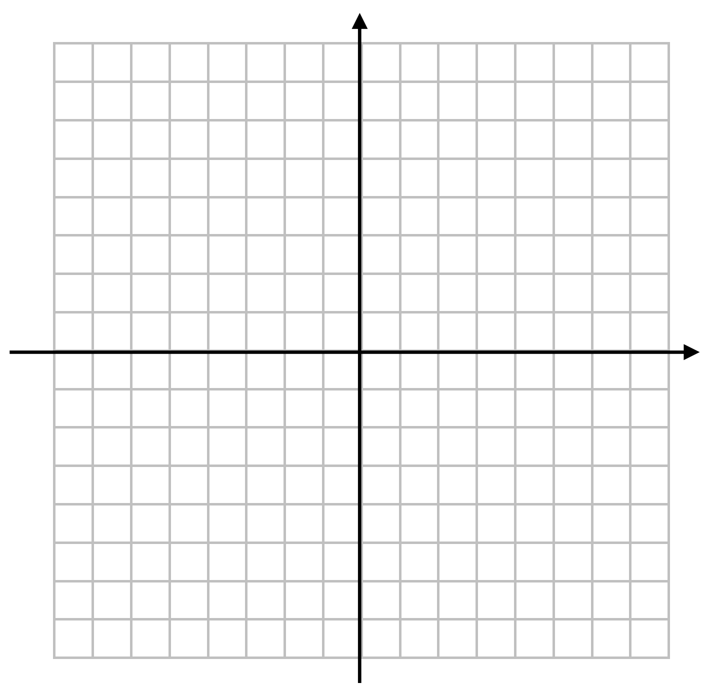


How to write the equation of a circle in standard form:

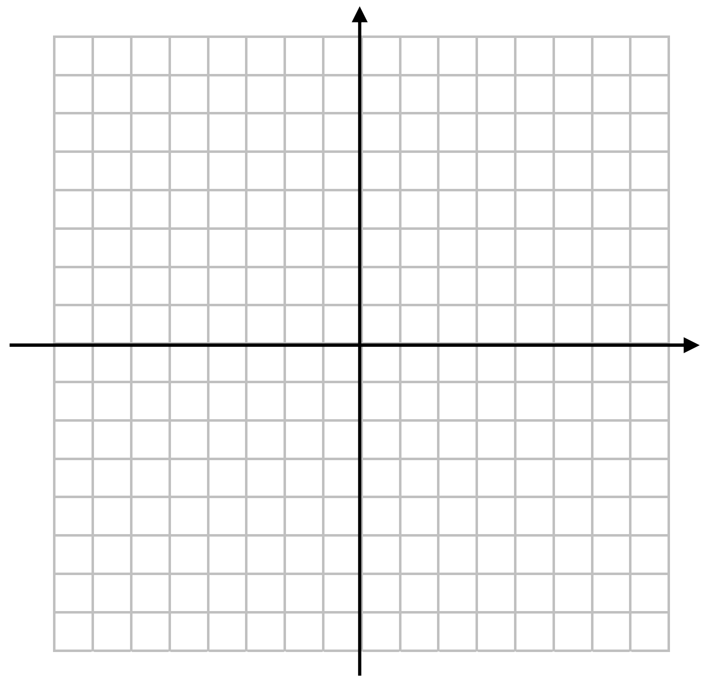
General (or expanded) form of a circle: $x^2+y^2+ax+by+c=0$

To find the center and radius of a circle written in expanded form, we need to *complete the square* (sometimes twice!).

Example 3: Find the center and radius of $x^2+y^2-4y-20=0$. Graph it.



Example 4: Find the center and radius of $x^2 + y^2 - 6x + 2y = 6$. Graph it.



Example 5: Write the circle $x^2 - 3x + y^2 + 7y - 2 = 0$ and find its center and radius.

Example 6: Write the circle $x^2 + y^2 + \frac{1}{2}x + \frac{1}{3}y - \frac{5}{6} = 0$ and find its center and radius.

Example 7: Write the equation of the circle with center $(-3,-5)$ and tangent to the *x*-axis.

Example 8: Write the equation of the circle such that the endpoints of a diameter are $(-2,7)$ and $(10,-9)$.