3.2: Polynomial Functions and Their Graphs

Definition: A polynomial function is a function which can be written in the form

$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$.

Example 1:

The numbers $a_0, a_1, a_2, ..., a_n$ are called the coefficients of the polynomial function.

Note: *The variable is only raised to positive integer powers–no negative or fractional exponents. However, the coefficients may be any real numbers, including fractions or irrational numbers like* π or $\sqrt{7}$.

The degree of the polynomial is the largest exponent on *x*. (Degree is usually denoted by *n*.)

The leading coefficient of a polynomial is the coefficient of the term with the largest power of *x*.

Example 2: $f(x) = 2x^2 - 9x^4 + 7x^3 - 12$

Degree:

Leading coefficient:

Facts about polynomials:

- They are smooth curves, with no jumps or sharp points.

Example 3:

- A polynomial has at most $n-1$ turning points.
- A polynomial has at most n x-intercepts.
- A polynomial has exactly one y-intercept.
- Every polynomial has domain $(-\infty, \infty)$.
- Near the ends,

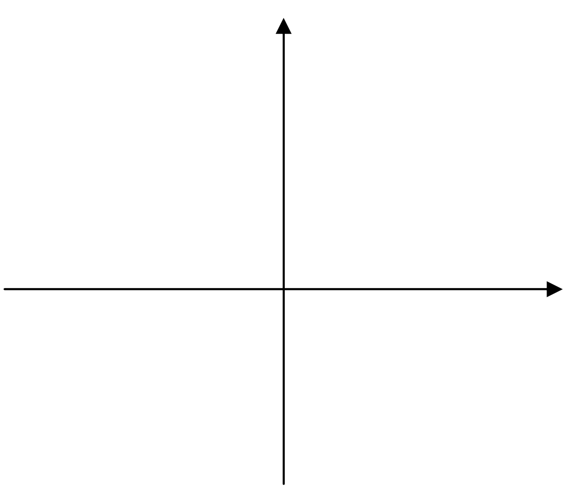
    Odd-degree polynomials look like $y = \pm x^3$.

    Even-degree polynomials look like $y = \pm x^2$.

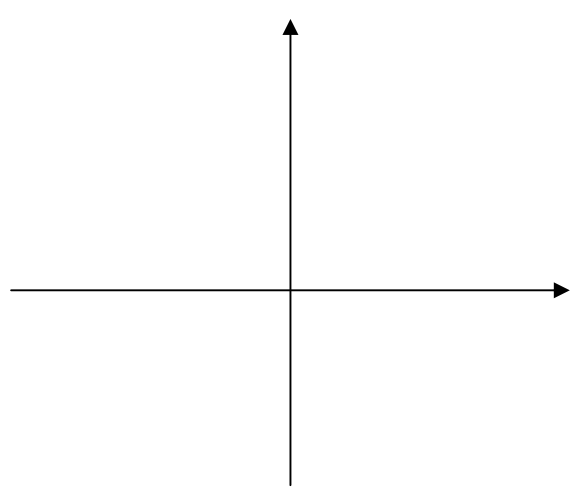
Power functions:

A *power function* is generally defined to be a polynomial which takes the form $f(x) = ax^n$, where n is a positive integer. Modifications of power functions can be graphed using transformations.

Even-degree power functions:

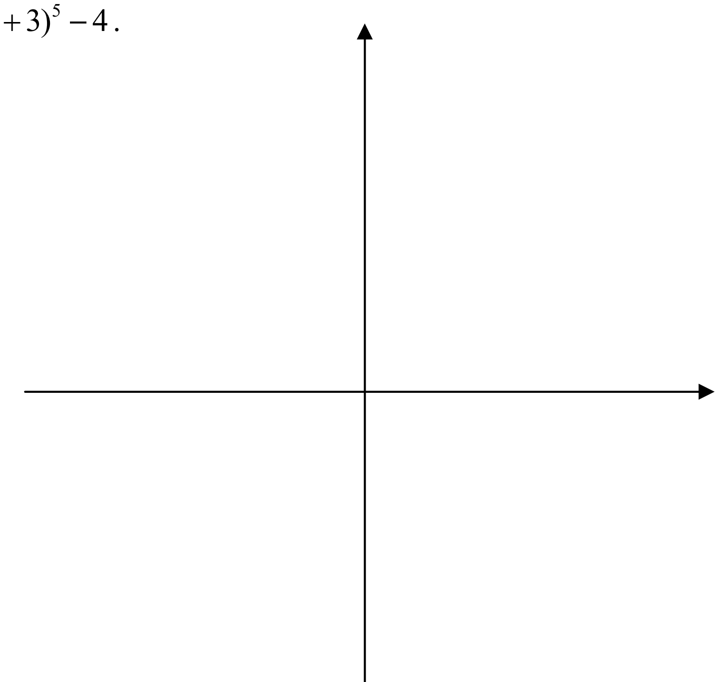


Odd-degree power functions:



Note: Multiplying any function by a will multiply all the y-values by a. The general shape will stay the same.

Example 4: Sketch the graph of $y = -(x+3)^5 - 4$.



Zeros of polynomials:

If f is a polynomial and c is a real number for which $f(c) = 0$, then c is called a *zero* of f, or a *root* of f.

If c is a zero of f, then
- c is an x-intercept of the graph of f.
- $(x-c)$ is a factor of f.

So if we have a polynomial in factored form, we know all of its x-intercepts.
- every factor gives us an x-intercept.
- every x-intercept gives us a factor.

Example 5: Consider the function $f(x) = 3x(x-3)^6(2x-1)^3(x+2)^2$.

Zeros (x-intercepts):

To get the degree, add the multiplicities of all the factors:

The leading term is:

<u>Steps to graphing other polynomials</u>:

1. Factor and find x-intercepts.

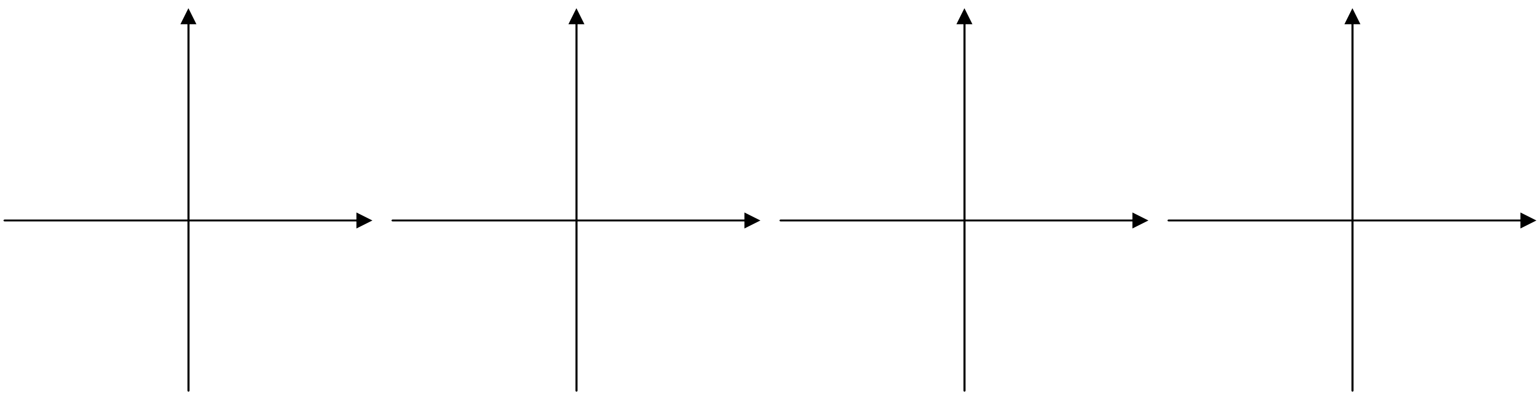
2. Mark x-intercepts on x-axis.

3. Determine the leading term.

   - Degree: is it odd or even?

   - Sign: is the coefficient positive or negative?

4. Determine the end behavior. What does it "look like"?



Even Degree Sign (+) | Even Degree Sign (-) | Odd Degree Sign (+) | Odd Degree Sign (-)

5. For each x-intercept, determine the behavior.

   - <u>Even multiplicity</u>: touches x-axis, but doesn't cross (looks like a parabola there).

   - <u>Odd multiplicity of 1</u>: crosses the x-axis (looks like a line there).

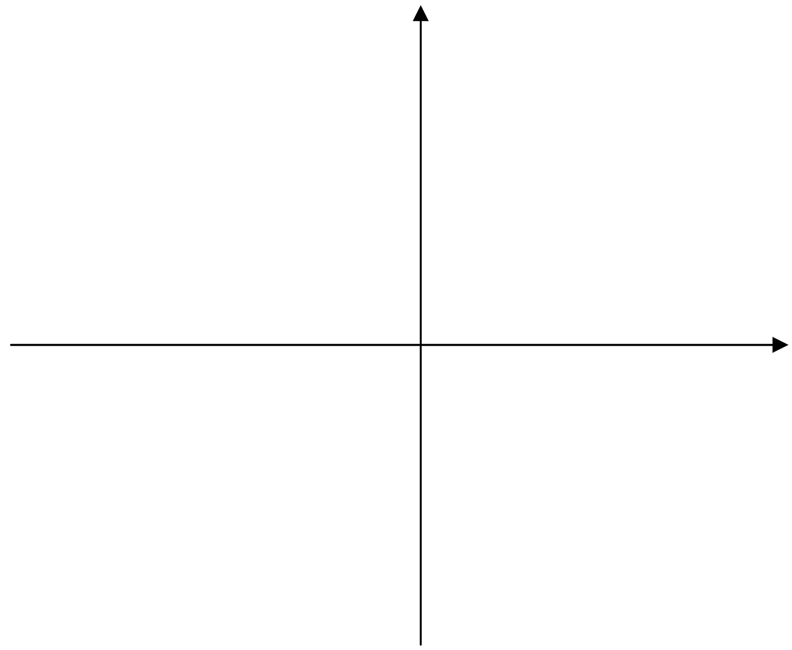
   - <u>Odd multiplicity</u> ≥ 3 : crosses the x-axis and looks like a cubic there.

<u>Note</u>: It helps to make a table as shown in the examples below.

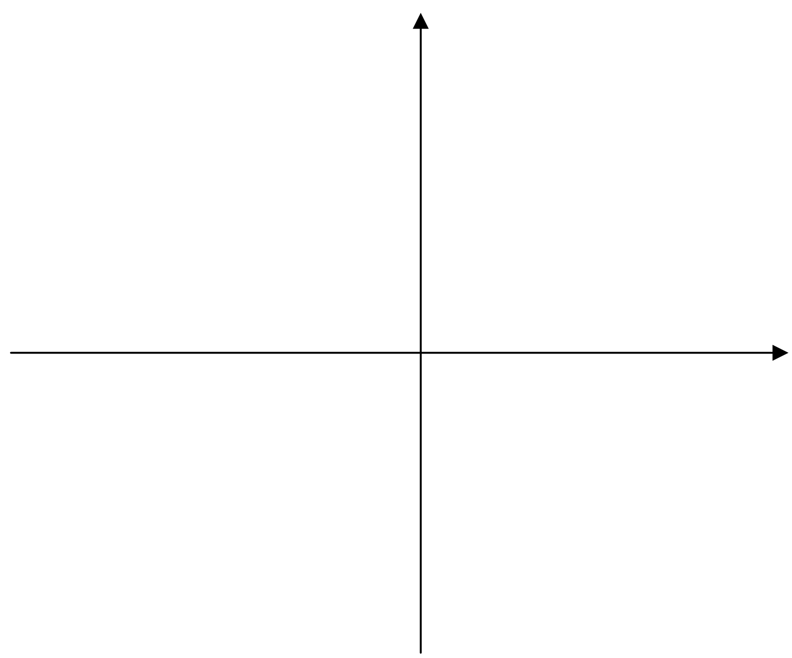
6. Draw the graph, being careful to make a nice smooth curve with no sharp corners.

<u>Note</u>: without calculus or plotting lots of points, we don't have enough information to know how high or how low the turning points are.

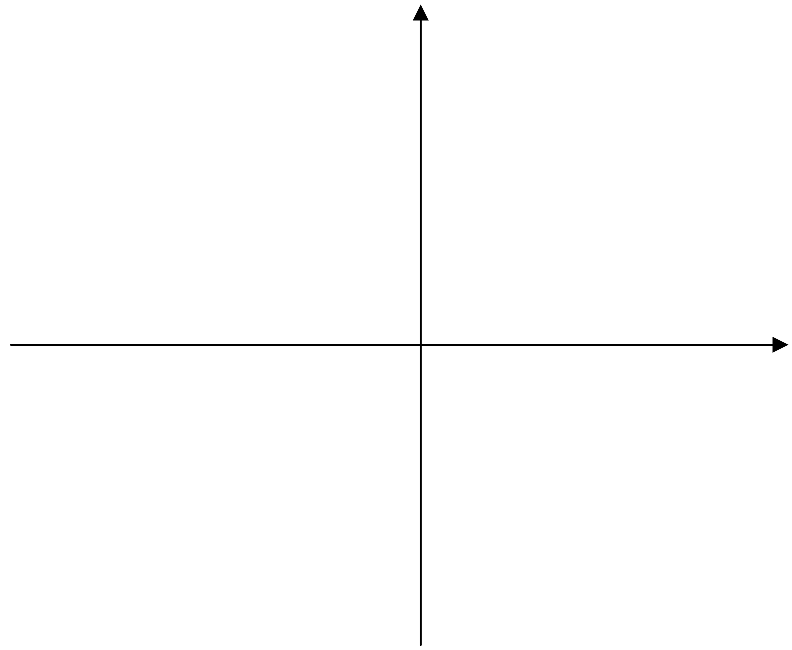
Example 6: Sketch the graph of $g(x) = -(x-1)(x+3)^3(x-4)^2$.



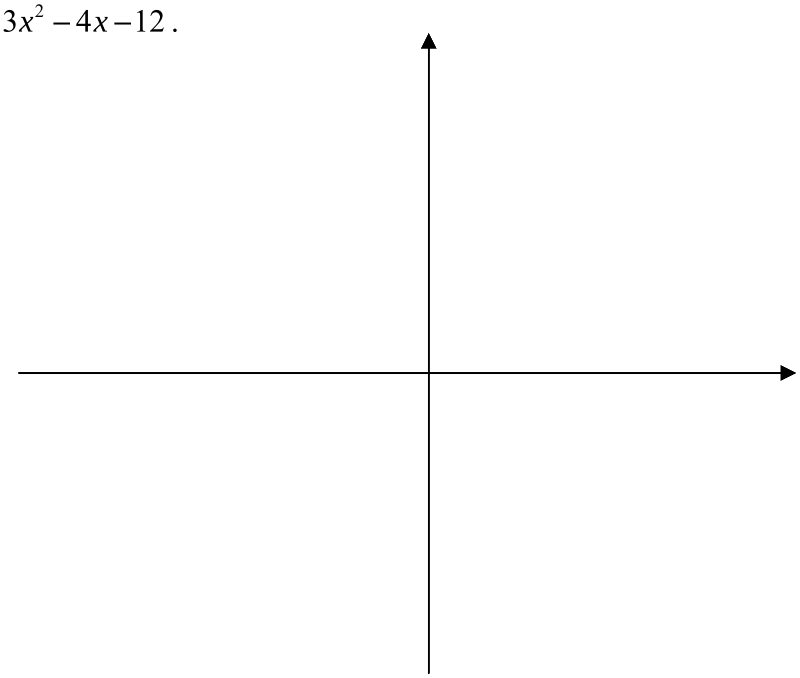
Example 7: Sketch the graph of $f(x) = x^3(4-x)(x+5)(x-8)^2$.



Example 8: Sketch the graph of $y = 2(x+1)^5(x+7)^2(2x-7)$.



Example 9: Sketch the graph of $P(x) = x^3 + 3x^2 - 4x - 12$.



<u>Intermediate Value Theorem for Polynomials</u>

Let f be a polynomial function with real coefficients. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of c between a and b for which $f(c) = 0$.

Example 1: Show that $f(x) = 3x^3 - 10x + 9$ has a real zero between -3 and -2.