3.3: Dividing Polynomials; Remainder and Factor Theorems

Terms you should know:

$$\begin{array}{r|l} & \text{Quotient} \\ \hline \text{Divisor} & \text{Dividend} \\ & ***** \\ & ***** \\ & \underline{*****} \\ & \text{Remainder} \end{array}$$

Recall: Dividend = (Divisor)(Quotient) + Remainder

So… $$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

Example 1: Divide $\dfrac{379}{12}$.

Long division of polynomials will be done similarly…

Example 2: Divide $\dfrac{3x^2+7x+9}{x+2}$.

Important:

You *must* put zeros in place of any missing powers of *x*.
The 0's act as "placeholders" just as they do in our number system.

For example, 206 is certainly not the same number as 26. The 0 "holds" the 10's place.

Similarly, when doing division, we write $2x^2 + 0x + 6$ instead of $2x^2 + 6$.

Example 3: Divide $\dfrac{8x^4 - 10x^2 + 3x - 2}{2x - 1}$.

Example 4: Divide $\dfrac{x^4 - 5x^3 + 6x - 8}{x^2 + 2}$.

Example 5: Divide $\dfrac{x^3 - 2x^2 - 5x + 6}{x - 3}$.

You can use this long division to factor $x^3 - 2x^2 - 5x + 6$…

The Factor Theorem

Let $f(x)$ be a polynomial,

- If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
- If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

In other words, c is a zero (root, x-intercept) of $f(x)$ if and only if $x - c$ is a factor of $f(x)$.

Example 6: Show that 4 is a zero of the function $P(x) = x^3 + 3x^2 - 18x - 40$. Use this fact to factor P completely and find all other zeros of $P(x)$.

***Synthetic Division*:** This is a "shortcut" which can be used to divide a polynomial by $x-c$, where c is a real number. Synthetic division does *not* work for divisors like x^2+1, x^2+2x-7, etc.

Example 7: Divide $\dfrac{3x^3+4x^2-13}{x-2}$.

Do it with long division first:

With synthetic division, the above work can be reduced to:

$$\begin{array}{r|rrrr} 2 & 3 & 4 & 0 & -13 \\ & & 6 & 20 & 40 \\ \hline & 3 & 10 & 20 & 27 \end{array}$$

How to do synthetic division:

Step 1: Write the dividend in descending powers. Then, copy the coefficients and insert 0 for any missing powers.

$$\begin{array}{rrrr} 3 & 4 & 0 & -13 \end{array}$$

Step 2: Write the usual division symbol over the numbers. If the divisor is $x-c$, write "c" to the left. Leave space under the numbers and draw a horizontal line.

$$\begin{array}{r|rrrr} 2 & 3 & 4 & 0 & -13 \\ & & & & \\ \hline & & & & \end{array}$$

Step 3: Bring down the first number under the division sign.

$$\begin{array}{r|rrrr} 2 & 3 & 4 & 0 & -13 \\ & & & & \\ \hline & 3 & & & \end{array}$$

Step 4: Multiply the latest entry of Row 3 by the number "outside" and write that product on Row 2 in the next column.

$$\begin{array}{r|rrrr} 2 & 3 & 4 & 0 & -13 \\ & & 6 & & \\ \hline & 3 & & & \end{array}$$

Step 5: Add the numbers in the new column and write the sum on Row 3.

$$\begin{array}{r|rrrr} 2 & 3 & 4 & 0 & -13 \\ & & 6 & & \\ \hline & 3 & 10 & & \end{array}$$

Step 6: Repeat Step 4 and Step 5 until all the columns have been used.

$$\begin{array}{r|rrrr} 2 & 3 & 4 & 0 & -13 \\ & & 6 & 20 & \\ \hline & 3 & 10 & 20 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 3 & 4 & 0 & -13 \\ & & 6 & 20 & 40 \\ \hline & 3 & 10 & 20 & 27 \end{array}$$

Step 7: The rightmost number on Row 3 is the remainder. The remaining entries on Row 3 are the coefficients of the quotient. The degree of the quotient is one less than the degree of the dividend.

Remainder: 27

Quotient: $3x^2+10x+20$

Therefore,

$$3x^3+4x^2-13=(x-2)(3x^2+10x+20)+\frac{27}{x-2}$$

Example 8: Divide $\dfrac{x^3-2x^2-5x+6}{x-3}$.

Example 9: Divide $\dfrac{x^4+7x^3+8x^2-28x-40}{x+3}$.

Example 10: Divide $\dfrac{x^3+3x^2-18x-40}{x-4}$.

Example 11: Show that 3 is a zero of the function $P(x) = 2x^3 + 7x^2 - 19x - 60$. Use this fact to factor P completely and find all other zeros of $P(x)$.

Remainder Theorem:
If the polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Example 12: Given $f(x) = x^4 + 9x^3 + 5x^2 - 12x - 13$, use the Remainder Theorem to find $f(-4)$.

Example 13: Solve the equation $x^4 + 7x^3 + 8x^2 - 28x - 48 = 0$ given that -3 is a zero of $P(x) = x^4 + 7x^3 + 8x^2 - 28x - 48$.

Example: Divide $\dfrac{6x^3-19x^2-65x+50}{2x+5}$.

Step 1: Divide $\dfrac{6x^3}{2x}=$

$$
\begin{array}{r}
3x^2 \\
2x+5\overline{)6x^3-19x^2-65x+50}
\end{array}
$$

Step 2: Multiply $3x^2$ by $2x+5$

$$
\begin{array}{r}
3x^2 \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
\underline{6x^3+15x^2}
\end{array}
$$

Step 3: Subtract (Don't forget the parentheses!)

$$
\begin{array}{r}
3x^2 \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
-\underline{(6x^3+15x^2)} \\
-34x^2
\end{array}
$$

Step 4: Bring Down

$$
\begin{array}{r}
3x^2 \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
-\underline{(6x^3+15x^2)} \\
-34x^2-65x
\end{array}
$$

Repeat the steps…

Step 1: Divide $\dfrac{-34x^2}{2x}=$

$$
\begin{array}{r}
3x^2-17x \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
-\underline{(6x^3+15x^2)} \\
-34x^2-65x
\end{array}
$$

Step 2: Multiply $-17x$ by $2x+5$

$$
\begin{array}{r}
3x^2-17x \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
-\underline{(6x^3+15x^2)} \\
-34x^2-65x \\
-34x^2-85x
\end{array}
$$

Go to the top right…

Step 4: Bring Down

$$
\begin{array}{r}
3x^2-17x \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
-\underline{(6x^3+15x^2)} \\
-34x^2-65x \\
-\underline{(-34x^2-85x)} \\
20x+50
\end{array}
$$

Repeat the steps…

Step 1: Divide $\dfrac{20x}{2x}=$

$$
\begin{array}{r}
3x^2-17x+10 \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
-\underline{(6x^3+15x^2)} \\
-34x^2-65x \\
-\underline{(-34x^2-85x)} \\
20x+50
\end{array}
$$

Step 2: Multiply 10 by $2x+5$

$$
\begin{array}{r}
3x^2-17x+10 \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
-\underline{(6x^3+15x^2)} \\
-34x^2-65x \\
-\underline{(-34x^2-85x)} \\
20x+50 \\
20x+50
\end{array}
$$

Step 3: Subtract (Don't forget the parentheses!)

$$
\begin{array}{r}
3x^2-17x+10 \\
2x+5\overline{)6x^3-19x^2-65x+50} \\
-\underline{(6x^3+15x^2)} \\
-34x^2-65x \\
-\underline{(-34x^2-85x)} \\
20x+50 \\
-\underline{(20x+50)} \\
\text{Remainder} \rightarrow 0
\end{array}
$$

Extra Handout on Polynomial Long Division