3.5: Rational Functions

Definition: A *rational* function is a function that can be written in the form $R(x)=\frac{f(x)}{g(x)}$, where f and g are polynomials.

Recall: Rational number: can be written as a ratio (fraction, quotient) of 2 integers.

Example 1:

Examples of Rational Functions:

$$f(x)=\frac{x^2+9}{x^3-3x+7}$$

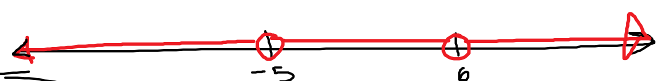
$$g(x)=\frac{x}{x+5},\quad h(x)=\frac{3}{x+5},\quad j(x)=\frac{x^2-2x+1}{x^3-5}$$

The domain of the rational function $R(x)=\frac{f(x)}{g(x)}$ consists of all real numbers x such that $g(x)\neq 0$.

Shorthand: degree of f = deg(f).

Example 2: Find the domain of $f(x)=\frac{x^2+7x+10}{x^2-x-30}$. $=\frac{(x+2)(x+5)}{(x-6)(x+5)}$

Domain: $x\neq 6,\ x\neq -5$

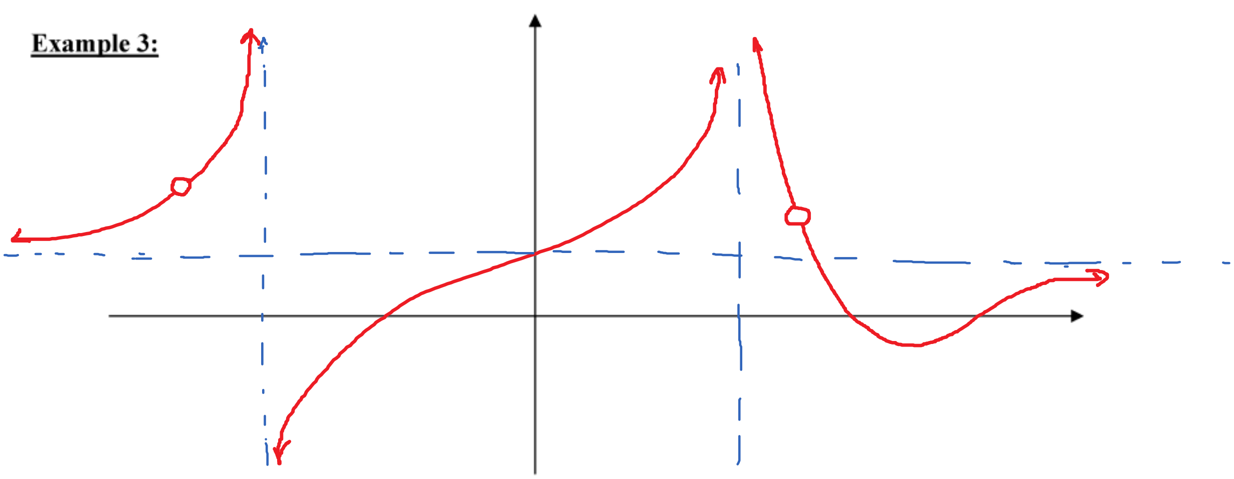


Domain: $(-\infty,-5)\cup(-5,6)\cup(6,\infty)$

Graphs of Rational Functions

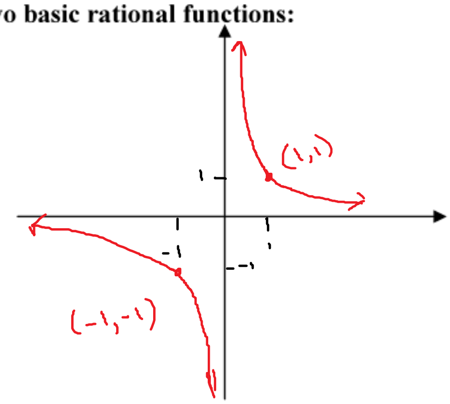
The graphs of rational functions are characterized by asymptotes and/or holes.

Example 3:

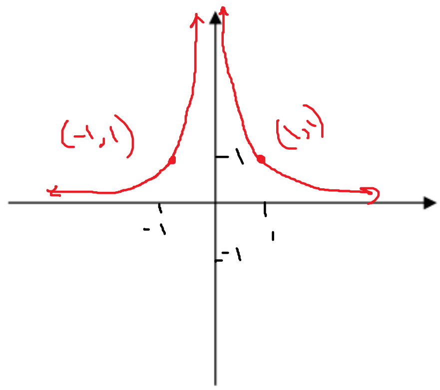


Know these but don't worry about transformations of them.

Two basic rational functions:



$f(x) = \frac{1}{x}$

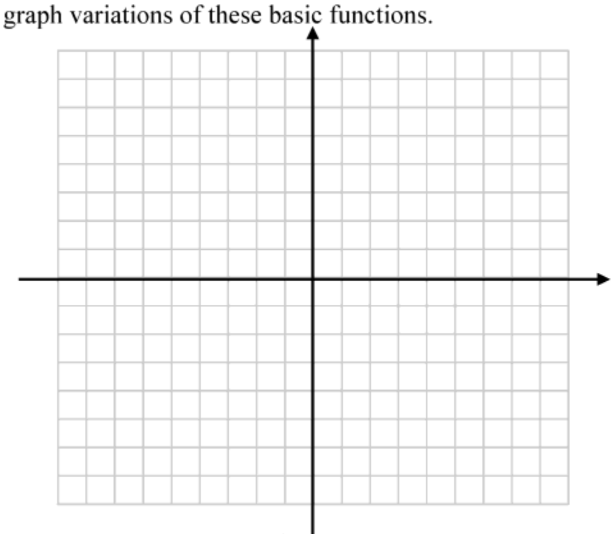


$f(x) = \frac{1}{x^2}$

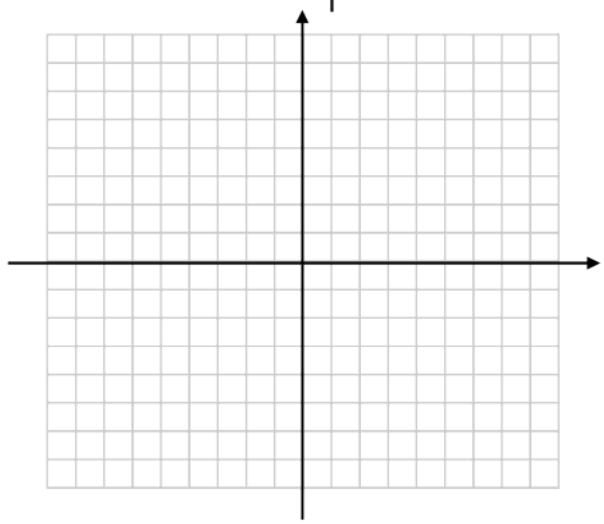
We can use the transformations we already know to graph variations of these basic functions.

__**Example 4:**__ Graph $f(x) = \frac{1}{x+2} - 3$.

left 2, down 3



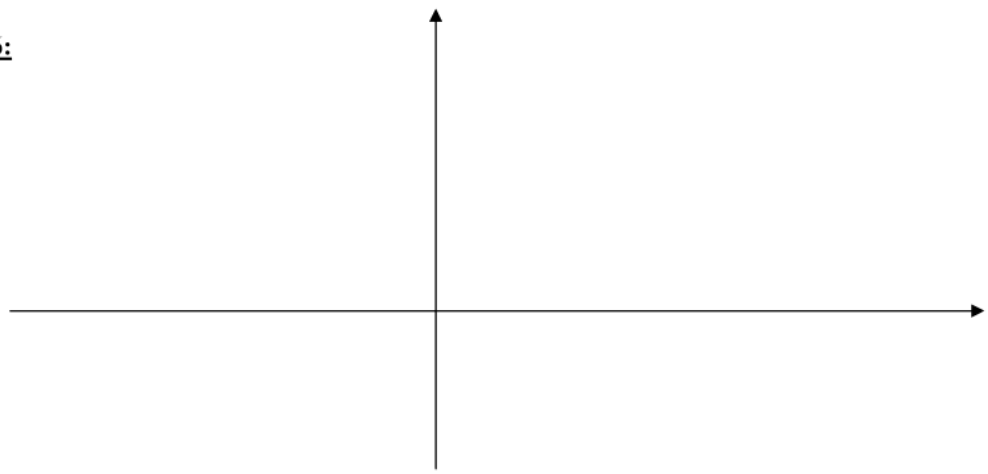
__**Example 5:**__ Graph $f(x) = -\frac{1}{(x-3)^2}$.



Some useful notation:

$x \to a^+$ means that x approaches a from the right.
$x \to a^-$ means that x approaches a from the left.
$x \to \infty$ means that x approaches infinity (increases without bound).
$x \to -\infty$ means that x approaches negative infinity (decreases without bound).

Example 6:



Know how to find these!

To find x-intercepts, vertical asymptotes and holes:

Factor the numerator and denominator.

- If a factor "cancels" (appears in both the numerator and denominator), then there is a hole where that factor equals zero.
- If a factor in the denominator doesn't "cancel", then there is a vertical asymptote where that factor equals zero.
- If a factor in the numerator doesn't "cancel", then there is an x-intercept where that factor equals zero.

Denominator $\Rightarrow$ vertical asymptotes

Numerator $\Rightarrow$ x-intercepts

Example 7: Find x-intercepts, vertical asymptotes and holes for $f(x)=\dfrac{x^2-3x-10}{x^2-x-6}$.

$$f(x)=\frac{x^2-3x-10}{x^2-x-6}=\frac{(x-5)(x+2)}{(x-3)(x+2)}$$

Find undefined places: $x \neq 3,\ x \neq -2$

"Canceled" version: $g(x)=\dfrac{x-5}{x-3}$

x-intercept of $f(x)$: 5
Hole at $x=-2$
Vertical asymptote of $f(x)$: $x=3$

Note: $\dfrac{0}{0} \neq 1$

g(x) has x-intercept 5 and vertical asymptote x=3

Domain of $g(x)$: $x \neq 3$

To find the x-intercept of $g(x)=\dfrac{x-5}{x-3}$

Set $y=0$: $0=\dfrac{x-5}{x-3}$

$0=x-5$

$x=5$

x-intercept is 5 or $(5,0)$

Example 8: Find x-intercepts, vertical asymptotes and holes for $f(x)=\dfrac{2x^2+5x-3}{x^3-3x^2-4x}$.

$$f(x)=\frac{2x^2+5x-3}{x^3-3x^2-4x}=\frac{(2x-1)(x+3)}{x(x^2-3x-4)}$$

$$=\frac{(2x-1)(x+3)}{x(x-4)(x+1)}$$

x-intercepts: $\frac{1}{2}$, -3 or $(\frac{1}{2},0)$, $(-3,0)$
Vertical asymptotes: $x=0$, $x=4$, $x=-1$
Holes: none

Scratch
$2x-1=0$
$2x=1$
$x=\frac{1}{2}$

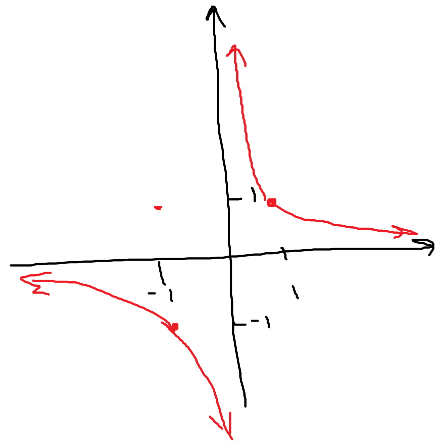
Example. function with a hole.

Graph $f(x) = \frac{x-3}{x^2-3x}$

Factor: $f(x) = \frac{x-3}{x(x-3)}$

"Cancelled" version: $g(x) = \frac{1}{x}$

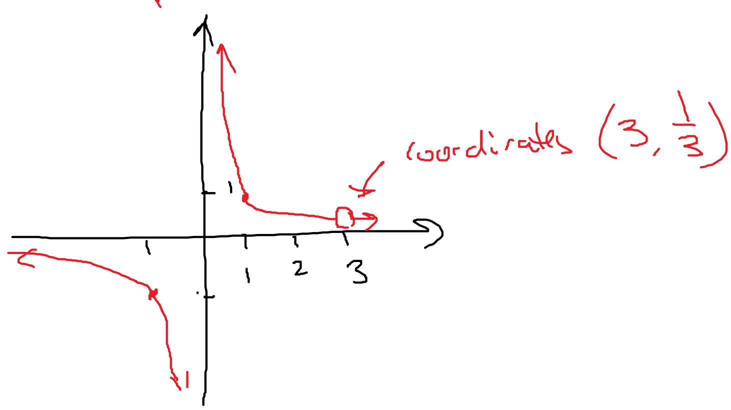
Graph of $g(x) = \frac{1}{x}$



Graph $f(x) = \frac{x-3}{x(x-3)}$

Domain

$x \neq 0, x \neq 3$



Finding Horizontal Asymptotes:

Horizontal asymptotes have to do with what happens to the *y*-values as *x* becomes very large or very small. If the *y*-values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

Case 1: deg(numerator) > deg(denominator) Horizontal Asymptote: None

If the degree of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote. In this case, the *y*-values get very "big" at the ends of the graph. The y-values do not approach a particular number.

$$f(x) = \frac{x^3 - 7x}{2x+5}$$ No horizontal asymptote

Case 2: deg(denominator) > deg(numerator) Horizontal Asymptote: $y = 0$

If the degree of the denominator is larger than the degree of the numerator, then the horizontal asymptote is the line $y = 0$ (the *x*-axis). In this case, the *y*-values get very small (approach 0) at the ends of the graph.

$$g(x) = \frac{2x+1}{3x^2 - 7x + 2}$$ Horizontal asymptote: $y=0$

Case 2: deg(denominator) = deg(numerator) Horizontal Asymptote: $y = \frac{a}{b}$

If the degree of the denominator is equal to the degree of the denominator, then the horizontal asymptote is the line $y = \frac{a}{b}$, where *a* is the leading coefficient of the numerator and *b* is the leading coefficient of the denominator.In this case, the *y*-values approach $\frac{a}{b}$ at the ends of the graph.

$$h(x) = \frac{2x^2 - 6x + 5}{6x^2 - 7}$$ as $x \to \pm\infty$, $y \to \frac{2x^2}{6x^2}$

$y \to \frac{2}{6} = \frac{1}{3}$

Horizontal Asymptote: $y = \frac{1}{3}$

<u>Summary:</u> Horizontal Asymptotes

1) deg(numerator) > deg(denominator) — <u>Horizontal Asymptote</u>: None

2) deg(denominator) > deg(numerator) — <u>Horizontal Asymptote</u>: $y = 0$

3) deg(denominator) = deg(numerator) — <u>Horizontal Asymptote</u>: $y = \frac{a}{b}$, where

- a is leading coefficient of numerator.
- b is leading coefficient of denominator.

Example 9: Determine the horizontal asymptote, if any, of $R(x) = \dfrac{x^9 + 3x^2}{x^6 + 4x^5 - 5}$.

No horizontal asymptote.

Example 10: Determine the horizontal asymptote, if any, of $f(x) = \dfrac{x^2 + 4x}{5x - x^3}$.

Degree of numerator is 2; degree of denominator is 3; So horizontal asymptote is y=0.

Example 11: Determine the horizontal asymptote, if any, of $g(x) = \dfrac{3x^2 + 5x}{4x^2 - 1}$.

y= 3/4

Example 12: Determine the horizontal asymptote, if any, of $g(x) = \dfrac{(x-2)(x+4)}{(x-1)(x+3)(x-5)}$.

Degree of numerator is 2; degree of denominator is 3; So horizontal asymptote is y=0.

Graphing rational functions:

1. Factor the numerator and denominator.

2. Find x-intercept(s) by setting numerator equal to zero.
   Note: if a factor "cancels", it results in a hole instead of an x-intercept.

3. Find vertical asymptotes (if any) by setting the denominator equal to zero.
   Remember: if a factor in the denominator "cancels", it results in a hole instead of a vertical asymptote.

4. Find y-intercept (if any) by substituting $x = 0$ into the original form of the function.
   Note: this is easier if you use the unfactored form.

5. Find horizontal asymptote (if any). There can be at most one horizontal asymptote.

6. Use the x-intercepts and vertical asymptotes to divide the x-axis into intervals.

7. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.

8. Graph! *Except for the breaks at the vertical asymptotes, the graph should be a nice smooth curve with no sharp corners.*

Note: It is possible for the graph to cross the horizontal asymptote, maybe even more than once. To figure out whether it crosses (and where), set y equal to the y-value of the horizontal asymptote and then solve for x.

How to handle holes:

"Cancel" the factor(s) that appear(s) in both the numerator and denominator. Graph the cancelled version of the function using the procedure above. Then, using your eraser, put an open circle at the x-value where the hole(s) should be.

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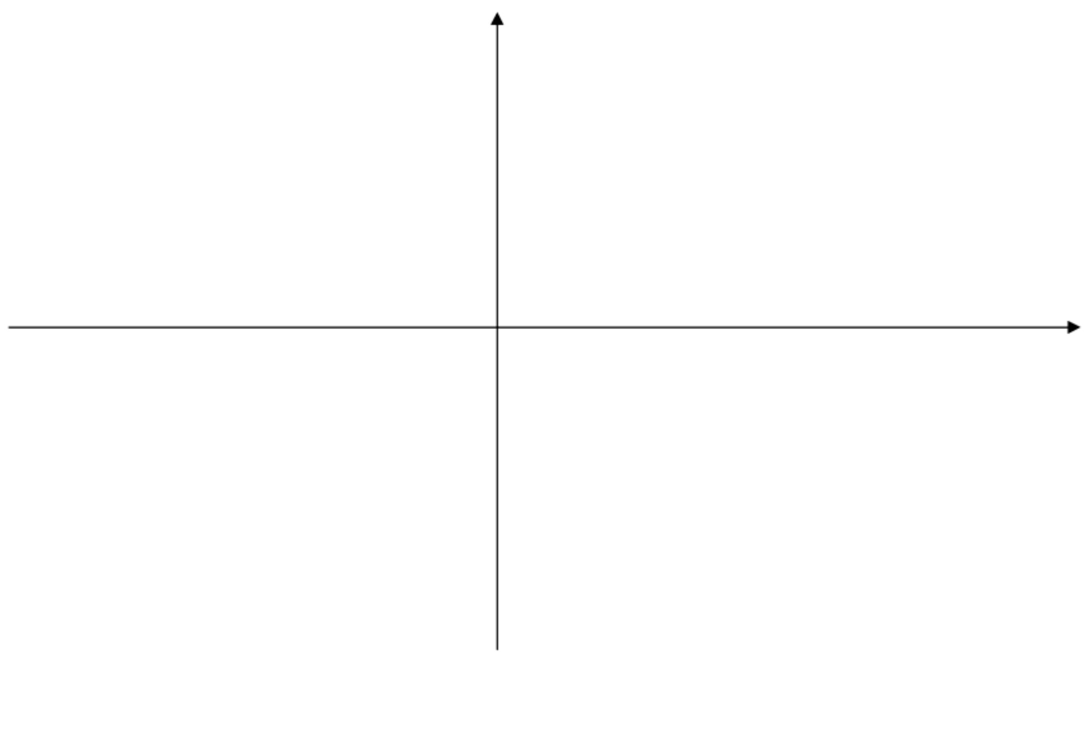
Example 13: Sketch the graph of $f(x)=\dfrac{x-3}{x^2-4}$.



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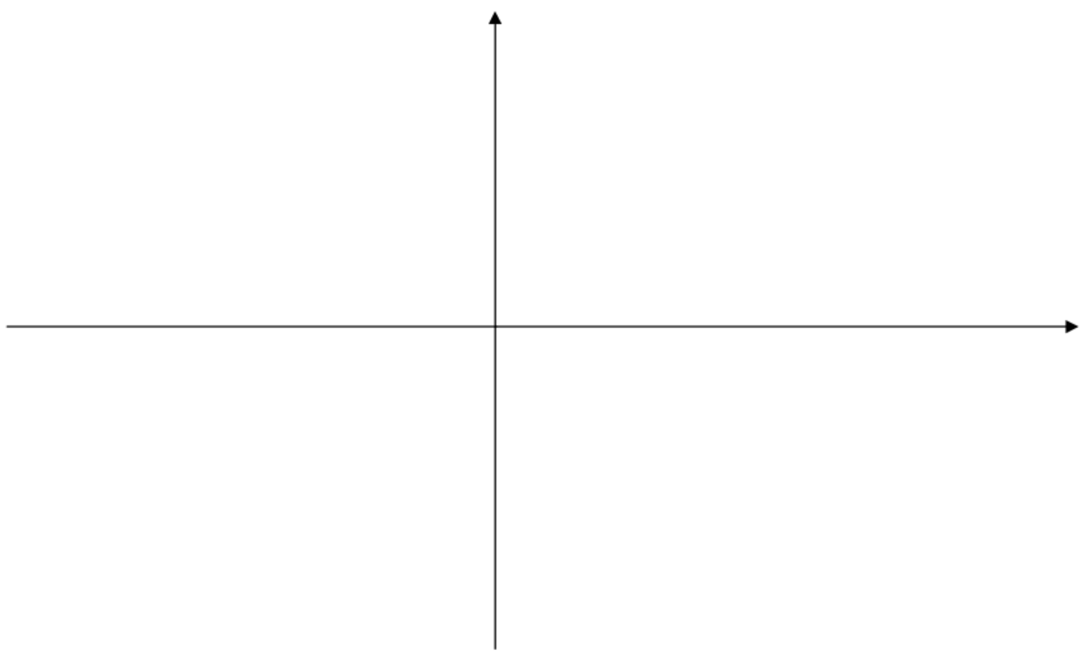
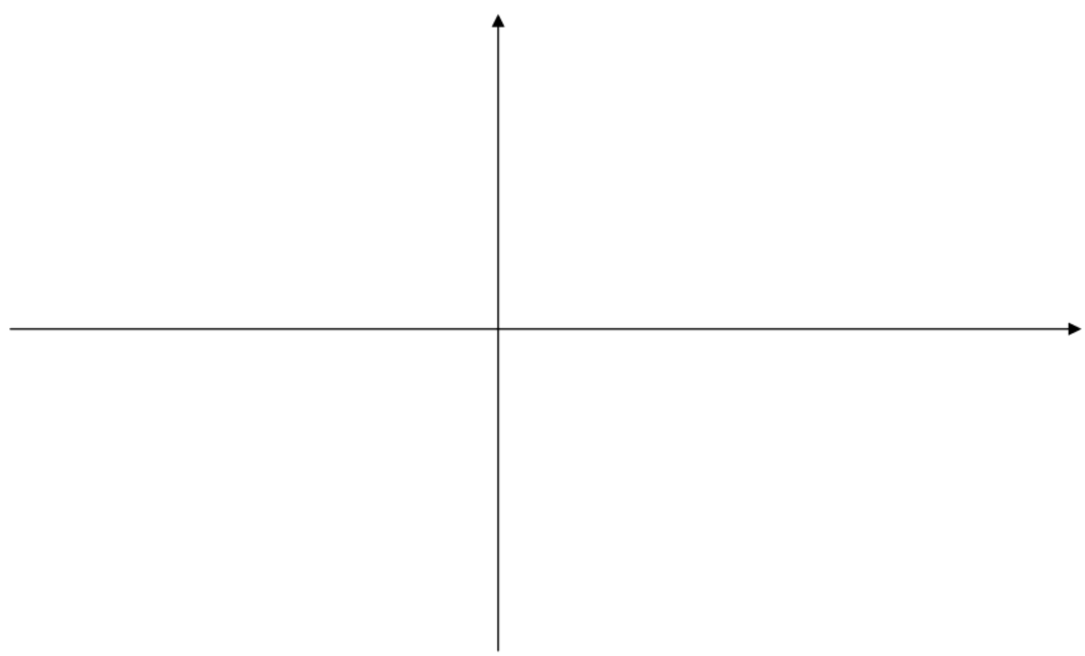
Example 14: Sketch the graph of $f(x) = \frac{x+5}{x-3}$.



Example 15: Sketch the graph of $f(x) = \frac{4x-12}{3-2x}$.



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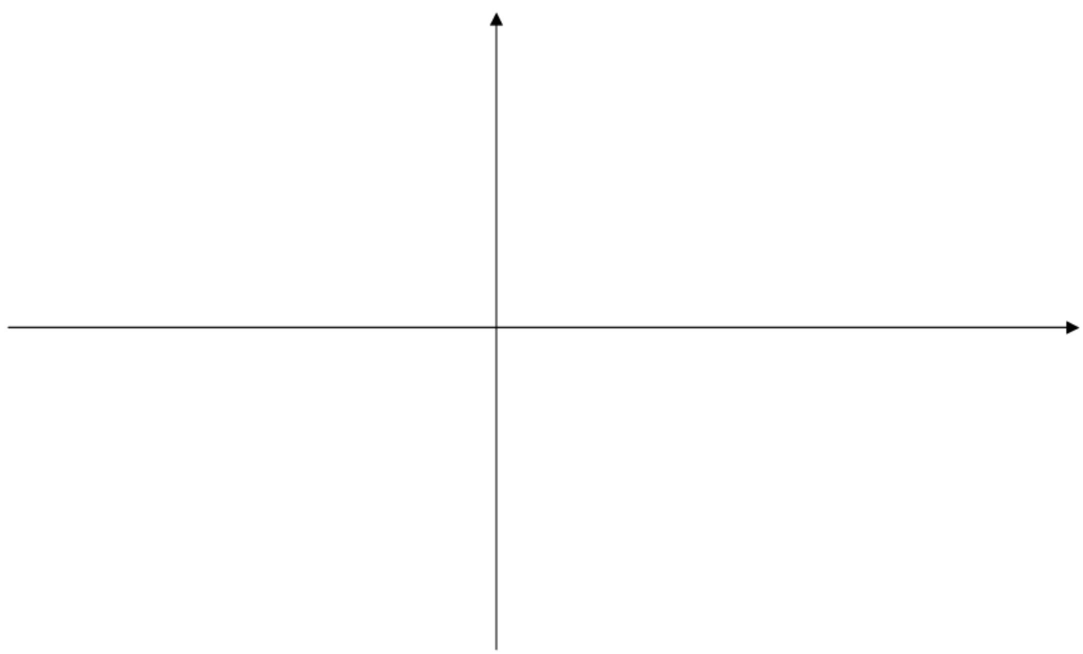
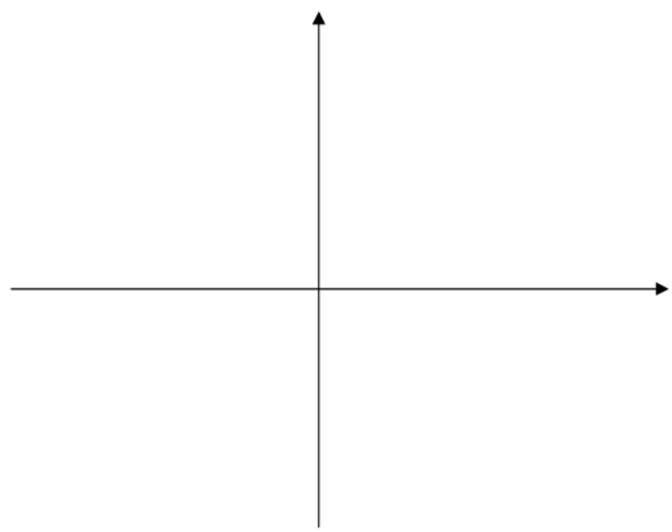
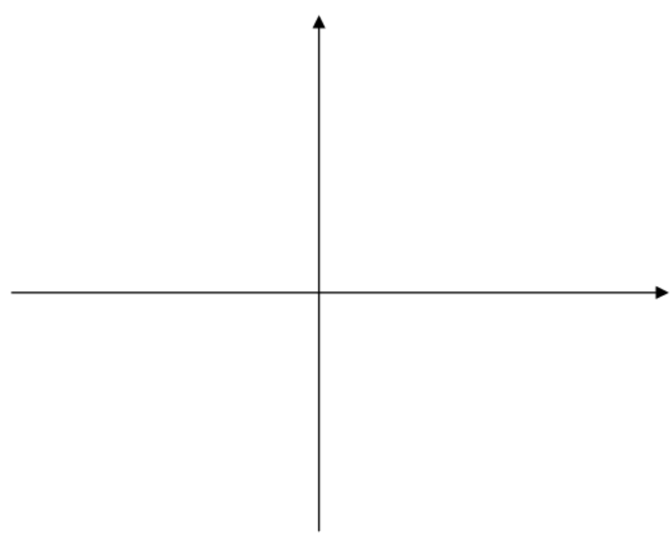
Example 16: Sketch the graph of $f(x)=\dfrac{4}{x^2-5x-6}$.



Example 17: Sketch the graph of $f(x)=\dfrac{x-2}{x^2-4}$.



Example 18: Sketch the graph of $f(x)=\dfrac{4x^2-13x+3}{x-3}$.



Example 19: Sketch the graph of $f(x) = \dfrac{-(x+4)(x-1)}{(x-4)(x+3)}$.