1314-4-2-Notes-logarithms

Thursday, October 17, 2019 11:22 AM

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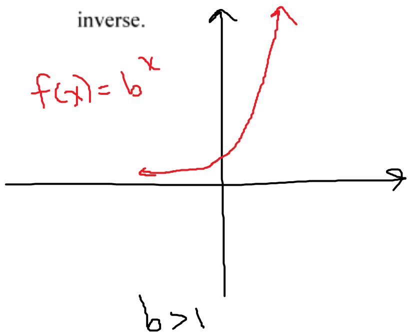
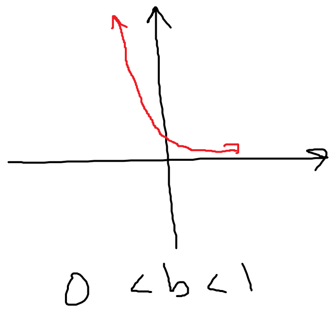
4.2: Logarithmic Functions

Looking at the graph of $f(x)=b^x$, we can see it is one-to-one. So every exponential function has an inverse.



both pass horizontal line test



Example 1: Consider the function $f(x)=3^x$.

x	$f(x)=3^x$
1	$3^1=3$
2	$3^2=9$
3	$3^3=27$
4	$3^4=81$

Then $f(1)=3$, $f(2)=9$, $f(3)=27$, and $f(4)=81$. This function has an inverse.

So $f^{-1}(3)=1$, $f^{-1}(9)=2$, $f^{-1}(27)=3$, and $f^{-1}(81)=4$.

We call this inverse function a *logarithmic function* and denote it $f^{-1}(x)=\log_3 x$.

So $f^{-1}(3)=\log_3 3=1$ and $f^{-1}(9)=\log_3 9=2$. Also $\log_3 27=3$ and $\log_3 81=4$.

x	$f^{-1}(x)=\log_3(x)=\log_3 x$
3	1
9	2
27	3
81	4

$\Rightarrow$ $\log_3(3)=1$
$\log_3(9)=2$
$\log_3(27)=3$

Every exponential function has an inverse. The inverses of exponential functions are called *logarithmic functions* (logarithms or logs for short).

> Definition: $\log_b x=y$ means $b^y=x$.
>
> The functions $f(x)=b^x$ and $g(x)=\log_b x$ are inverses of each other.
> b is called the *base* of the logarithm. Note: $b>0$, $b\neq 1$

Evaluating logarithms:

Example 2: Evaluate $\log_2 8$.

In other words, $\log_2 8$ is a number. What number is it?

This question is asking us to find a certain exponent. Specifically, "what exponent must I put on the 2 to give me 8?"

Said another way, "2 raised to what power is 8?"

$\log_2 8 = ?$

$2^{?} = 8$

answer: 3

$\boxed{\log_2 8 = 3}$ because $2^3 = 8$

Examples:

$\log_2 32 = \underline{5}$

$2^{?} = 32$

$\log_5 1 = \underline{0}$

$5^{?} = 1$

$\log_{10} 100 = \boxed{2}$

$10^{?} = 100$

$\log_3 \sqrt{3} = \boxed{\frac{1}{2}}$

$3^{?} = \sqrt{3} = \sqrt[2]{3}$

$\log_{\frac{1}{2}} 1 = \boxed{0}$

$\left(\frac{1}{2}\right)^{?} = 1$

$\log_3 \sqrt{3} = \boxed{\frac{1}{2}}$ (again)

$3^{?} = \sqrt{3}$

$\boxed{\begin{array}{l}\textbf{Recall:}\\ x^{\frac{1}{n}} = \sqrt[n]{x}\\ x^{-n} = \frac{1}{x^n}\end{array}}$

$\log_5\left(\sqrt[4]{5}\right) = \underline{\frac{1}{4}}$

$5^{?} = \sqrt[4]{5} = 5^{\frac{1}{4}}$

$\log_3\left(\frac{1}{9}\right) = \underline{-2}$

$3^{?} = \frac{1}{9}$

Note: $3^{?} = \frac{1}{9} = \frac{1}{3^2} = \frac{3^{-2}}{1} = 3^{-2}$

$\log_{10}\left(\frac{1}{1000}\right) = \boxed{-3}$

$10^{?} = \frac{1}{1000}$

Note: $10^{?} = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$

$\log_2\left(\sqrt[3]{16}\right) = \boxed{\frac{4}{3}}$

$2^{?} = \sqrt[3]{16}$

Note: $2^{?} = \sqrt[3]{16} = \sqrt[3]{2^4} = (2^4)^{1/3} = 2^{\frac{4}{3}}$

$\log_3\left(\frac{1}{\sqrt{3}}\right) = \boxed{-\frac{1}{2}}$

$3^{?} = \frac{1}{\sqrt{3}}$

Note: $3^{?} = \frac{1}{\sqrt{3}} = \frac{1}{3^{\frac{1}{2}}} = \frac{3^{-\frac{1}{2}}}{1} = 3^{-\frac{1}{2}}$

$\log_{64} 8 = \boxed{\frac{1}{2}}$

$64^{?} = 8$

Note: $8 = \sqrt{64} = 64^{\frac{1}{2}}$

Evaluating more complicated logarithms:

Example 3: $\log_4 32 = \boxed{\frac{5}{2}}$

$4^{?} = 32$

$4^{x} = 32$

Rewrite so that both sides are powers of the same base:

$4^x = 32$

$(2^2)^x = 2^5$

$2^{2x} = 2^5$

$2x = 5$

$\frac{2x}{2} = \frac{5}{2}$

$x = \frac{5}{2}$

Example 4: $\log_9\left(\frac{1}{27}\right) = \boxed{-\frac{3}{2}}$

$9^{?} = \frac{1}{27}$ Put in a variable: $9^x = \frac{1}{27}$

$(3^2)^x = \frac{1}{3^3}$

$3^{2x} = 3^{-3}$

$2x = -3$

$\frac{2x}{2} = \frac{-3}{2} \Rightarrow x = -\frac{3}{2}$

The natural logarithm:

Remember we said *e* was a very important number? It is so important that the logarithmic function of base *e* has its own special notation and its own button on your calculator.

The logarithm of base *e* is called the *natural logarithm*, which is abbreviated "ln".

$$\boxed{\log_e x = \ln x}$$

$\log_e(x) = \ln(x)$

Example 5: $\ln e^4 = \underline{\ 4\ }$ means $\log_e e^4 = \underline{\ 4\ }$

$e^{?} = e^4$ answer: 4

Recall: $\log_2 16 = \underline{\quad}$

$2^{?} = 16$ answer: 4

so $\log_2 16 = 4$

Example 6: $\ln\left(\frac{1}{e^3}\right) = \underline{\ -3\ }$ means $\log_e\left(\frac{1}{e^3}\right) = \underline{\ -3\ }$

$e^{?} = \frac{1}{e^3}$ Notice: $e^{?} = \frac{1}{e^3} = e^{-3}$

Example 7: Evaluate $\ln\sqrt{e}$. $\ln\sqrt{e} = \underline{\frac{1}{2}}$

Rewrite: $\log_e \sqrt{e} = \_\_\_\_$

$e^{?} = \sqrt{e}$ answer: $\frac{1}{2}$ because $e^{\frac{1}{2}} = \sqrt{e} \Rightarrow \boxed{\ln\sqrt{e} = \frac{1}{2}}$

$\log_e\sqrt{e} = \frac{1}{2}$

Example 8: Simplify $\ln\left(\frac{1}{\sqrt[3]{e^5}}\right)$.

$\ln\left(\frac{1}{\sqrt[3]{e^5}}\right) = \ln\left(\frac{1}{(e^5)^{1/3}}\right) = \ln\left(\frac{1}{e^{5/3}}\right) = \ln\left(e^{-\frac{5}{3}}\right) = \log_e\left(e^{-\frac{5}{3}}\right) = \boxed{-\frac{5}{3}}$

Note: $e^{?} = e^{-\frac{5}{3}}$ answer: $-\frac{5}{3}$

Example 9: Evaluate $\ln 1$.

$\ln 1 = \_\_\_\_$

$\log_e 1 = \underline{?} \Rightarrow e^{?} = 1$. We know $e^0 = 1$, so $\log_e 1 = 0$

so $\boxed{\ln(1) = 0}$

Example 10: Evaluate $\log_2(-4)$.

$\log_2(-4) = \_\_\_\_$

$2^{?} = -4$

It's impossible!

So $\boxed{\log_2(-4) \text{ is not defined}}$

Note: Try -2

$2^{-2} = \frac{1}{(2)^2} = \frac{1}{4}$

Try -1

$2^{-1} = \frac{1}{2}$

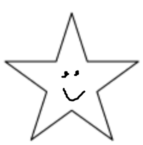
Try $\frac{1}{2}$: $2^{\frac{1}{2}} = \sqrt{2}$

Example 11: Evaluate $\log_5 0$.

$\log_5(0) = \_\_\_\_$

$5^{?} = 0$ This is impossible!

So $\boxed{\log_5(0) \text{ is not defined}}$



IMPORTANT:

You cannot apply a logarithm to zero or to a negative number!!!

Exponential and logarithmic forms for an equation:

Remember, $\log_b x = y$ means $b^y = x$.

$\log_b x = y \Rightarrow b^y = x$

Logarithmic form: $\log_b x = y$
Exponential form: $b^y = x$

Example 12: Convert each of the following to exponential form.

a) $\log_{10} 1000 = 3$

$\boxed{10^3 = 1000}$

b) $\log_a 178 = w$

$\boxed{a^w = 178}$

c) $\log_7 (y-3) = x$

$\boxed{7^x = y-3}$

d) $\log_x 6 = y^2 + 2$

$\boxed{x^{y^2+2} = 6}$

e) $\ln(w+3) = 5$

rewrite: $\log_e(w+3) = 5$

$\boxed{e^5 = w+3}$

Example 13: Convert each of the following to logarithmic form.

a) $7^x = 23$

$\boxed{\log_7(23) = x}$

Check: $\log_7 23 = x \Rightarrow 7^x = 23$

b) $y^{x-1} = 8$

$\boxed{\log_y 8 = x-1}$

Check $\log_y 8 = x-1 \Rightarrow y^{x-1} = 8$

c) $x^8 = u$

d) $(x-3)^2 = 6$

More about the relationship between $f(x)=b^x$ and $g(x)=\log_b x$:

Because $f(x)=b^x$ and $g(x)=\log_b x$ are inverses of one another, $f(g(x))=x$ and $g(f(x))=x$.

This gives us…

$$\boxed{\begin{array}{c}\log_b b^x = x \\ b^{\log_b x} = x\end{array}}$$

$\log_b(b^x) = x$

$b^{\log_b(x)} = x$

Example 14: Simplify $\log_3 3^{x+5}$.

$\log_3 3^{x+5} = \boxed{x+5}$

Example 15: Simplify $\log_2 2^{12}$.

$\log_2 2^{12} = \log_2(2^{12}) = \log_2(2)^{12} = \boxed{12}$

Example 16: Simplify $\ln e^{-2}$.

$\ln e^{-2} = \log_e e^{-2} = \boxed{-2}$

Example 17: Simplify $5^{\log_5 y}$.

$5^{\log_5 y} = \boxed{y}$

Example 18: Simplify $3^{\log_3 \sqrt{2}}$.

$3^{\log_3(\sqrt{2})} = \boxed{\sqrt{2}}$

Example 19: Simplify $e^{\ln(x^2+1)}$.

$e^{\ln(x^2+1)} = e^{\log_e(x^2+1)} = \boxed{x^2+1}$

The common logarithm:

Often $\log$ is used to mean $\log_{10}$. The logarithm of base 10 is called the *common logarithm*.

Example 20: Evaluate $\log\sqrt[3]{10}$.

For our context,

$\log\sqrt[3]{10} = \log_{10}\sqrt[3]{10}$

$= \log_{10} 10^{\frac{1}{3}} = \boxed{\frac{1}{3}}$

Evaluating logs on your calculator:

Example 21: Evaluate $\log_{10} 72$ on your calculator.

$10^{?} = 72$

$\log_{10}(72) = \log(72) \approx \boxed{1.8573}$

Example 22: Evaluate $\ln 12$ on your calculator.

$\ln(12) \approx \boxed{2.4849}$

Evaluate e^3 on your calculator

$e^3 \approx 20.086$

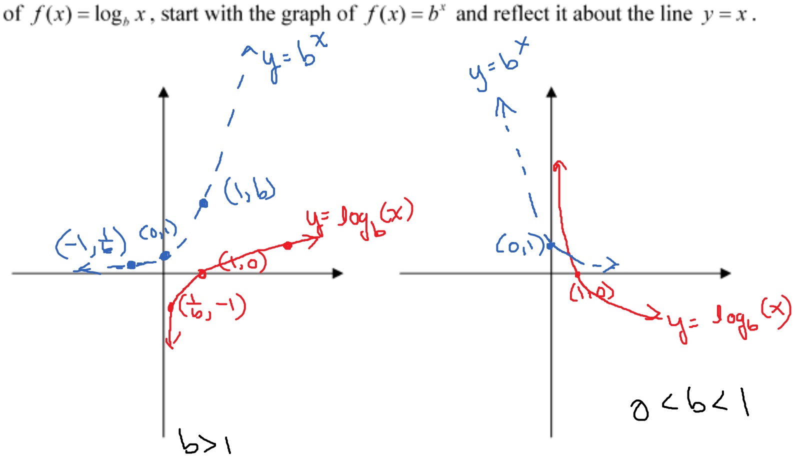
Graphs of logarithmic functions:

To get the graph of $f(x) = \log_b x$, start with the graph of $f(x) = b^x$ and reflect it about the line $y = x$. Why?

x	b^x
-1	$b^{-1} = \frac{1}{b}$
0	$b^0 = 1$
1	$b^1 = b$

reverse these:

x	$\log_b x$
$\frac{1}{b}$	-1
1	0
b	1



Facts about the graphs of $y = b^x$ and $y = \log_b x$:

$y = b^x$	$y = \log_b x$
Domain: $(-\infty, \infty)$	Domain: $(0, \infty)$
Range: $(0, \infty)$	Range: $(-\infty, \infty)$
Asymptote: The x-axis, so $y=0$	Asymptote: the y-axis, so $x=0$
Passes through: $(-1, \frac{1}{b})$, $(0,1)$, $(1,b)$	Passes through: $(\frac{1}{b}, -1)$, $(1,0)$, $(b,1)$

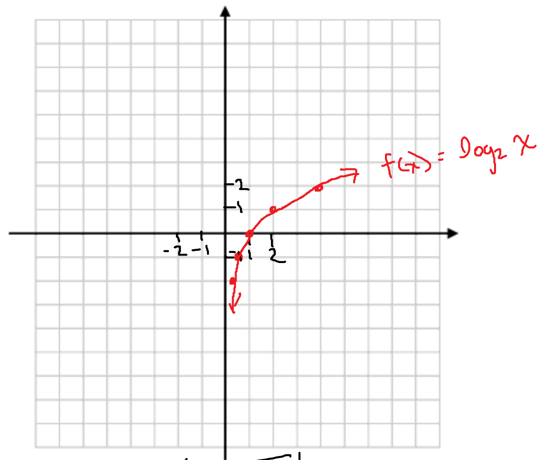
Example 23: Sketch the graph of $f(x) = \log_2 x$ by plotting points.

Start with $y = 2^x$

x	$y=2^x$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$

Use that to get pairs for $y = \log_2 x$

x	$\log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

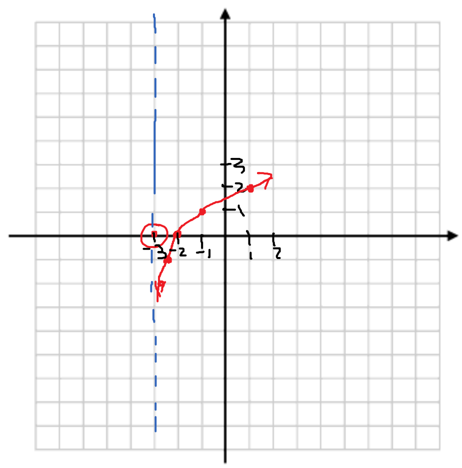


Vertical asymptote: $\boxed{x=0}$

Example 24: Graph $f(x) = \log_2(x+3)$.

Parent function: $y = \log_2(x)$
then shift it -3
(Note $x+3=0$
$x=-3$)

Use the same ordered pairs as previous example, but count from the point 3 units left of the origin



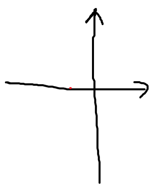
Asymptote: $\boxed{x = -3}$

Example 25: Graph $y = -\ln x$.

Parent function: $y = \ln x$

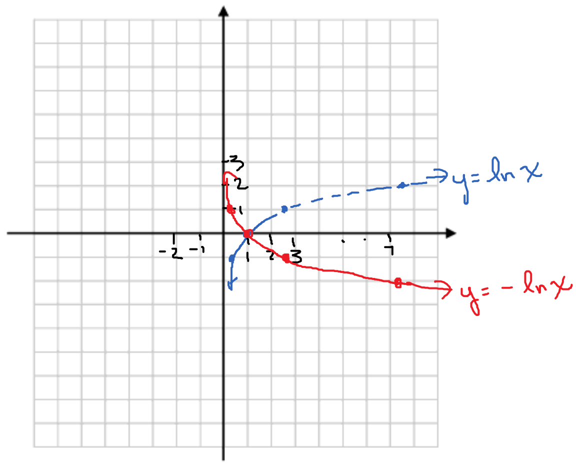
To graph $y = \ln x$, graph 1st $y = e^x$

x	$y = e^x$
-1	$e^{-1} = \frac{1}{e} \approx \frac{1}{3}$
0	$e^0 = 1$
1	$e^1 \approx 2.7$
2	$e^2 \approx 7.4$



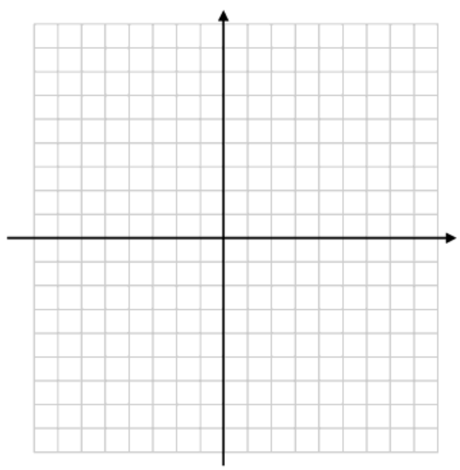
Reverse these to get $y = \ln x$

x	$y = \ln x$
$\frac{1}{3}$	-1
1	0
2.7	1
7.4	2

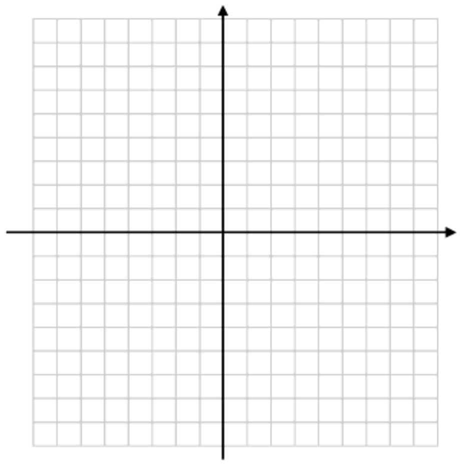


To get $y = -\ln x$, reflect it around x-axis

Example 26: Graph $g(x) = \ln(x+2) - 1$.



Example 27: Graph $f(x) = -2 - \log_2(x-3)$.



Important: When graphing logarithmic and exponential functions, ALWAYS label the reference point with its coordinates. Also label the asymptote.

Example 28: Find the function of the form $y = \log_a x$ whose graph includes the point $(64, 3)$.

Finding the domain of logarithmic functions:

Example 29: Find the domain of $f(x) = \log_3(x-4)$.

Example 30: Find the domain of $f(x) = \log_5(x^2)$.

Example 31: Find the domain of $f(x) = \ln(x^2 + 6)$.

Example 32: Find the domain of $g(x) = \ln(3 - 2x)$.

Example 33: Find the domain of $h(x) = \ln(-x)$.

Solving simple logarithmic equations:

Example 34: Solve for x.

$$\log_2(x-1) = 5$$

Example 35: Solve for x.

$$\log_x 7 = \frac{1}{2}$$