4.2: Logarithmic Functions

Looking at the graph of $f(x) = b^x$, we can see it is one-to-one. So every exponential function has an inverse.

Example 1: Consider the function $f(x) = 3^x$.

Then $f(1) = 3$, $f(2) = 9$, $f(3) = 27$, and $f(4) = 81$. This function has an inverse.

So $f^{-1}(3) = 1$, $f^{-1}(9) = 2$, $f^{-1}(27) = 3$, and $f^{-1}(81) = 4$.

We call this inverse function a *logarithmic function* and denote it $f^{-1}(x) = \log_3 x$.

So $f^{-1}(3) = \log_3 3 = 1$ and $f^{-1}(9) = \log_3 9 = 2$. Also $\log_3 27 = 3$ and $\log_3 81 = 4$.

Every exponential function has an inverse. The inverses of exponential functions are called *logarithmic functions* (logarithms or logs for short).

> Definition: $\log_b x = y$ means $b^y = x$.
>
> The functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other.
> b is called the *base* of the logarithm.

Evaluating logarithms:

<u>Example 2:</u> Evaluate $\log_2 8$.

In other words, $\log_2 8$ is a number. What number is it?

This question is asking us to find a certain exponent. Specifically, "what exponent must I put on the 2 to give me 8?"

Said another way, "2 raised to what power is 8?"

<u>Examples</u>:

$\log_2 32 =$ ________

$\log_5 1 =$ ________

$\log_{10} 100 =$ ________

$\log_3 \sqrt{3} =$ ________

$\log_{\frac{1}{2}} 1 =$ ________

$\log_3 \sqrt{3} =$ ________

$\log_5\left(\sqrt[4]{5}\right) =$ ________

$\log_3\left(\frac{1}{9}\right) =$ ________

$\log_{10}\left(\frac{1}{1000}\right) =$ ________

$\log_2\left(\sqrt[3]{16}\right) =$ ________

$\log_3\left(\frac{1}{\sqrt{3}}\right) =$ ________

$\log_{64} 8 =$ ________

Evaluating more complicated logarithms:

<u>Example 3:</u> $\log_4 32 =$ _______

<u>Example 4:</u> $\log_9\left(\frac{1}{27}\right) =$ _______

The natural logarithm:

Remember we said *e* was a very important number? It is so important that the logarithmic function of base *e* has its own special notation and its own button on your calculator.

The logarithm of base *e* is called the *natural logarithm*, which is abbreviated "ln".

$$\boxed{\log_e x = \ln x}$$

<u>Example 5:</u> $\ln e^4 =$ _______

<u>Example 6:</u> $\ln\left(\frac{1}{e^3}\right) =$ _______

<u>Example 7:</u> Evaluate $\ln\sqrt{e}$.

<u>Example 8:</u> Simplify $\ln\left(\frac{1}{\sqrt[3]{e^5}}\right)$.

<u>Example 9:</u> Evaluate $\ln 1$.

<u>Example 10:</u> Evaluate $\log_2(-4)$.

<u>Example 11:</u> Evaluate $\log_5 0$.

IMPORTANT:

You cannot apply a logarithm to zero or to a negative number!!!

Exponential and logarithmic forms for an equation:

Remember, $\log_b x = y$ means $b^y = x$.

Logarithmic form: $\log_b x = y$
Exponential form: $b^y = x$

Example 12: Convert each of the following to exponential form.

a) $\log_{10} 1000 = 3$

b) $\log_a 178 = w$

c) $\log_7 (y-3) = x$

d) $\log_x 6 = y^2 + 2$

Example 13: Convert each of the following to logarithmic form.

a) $7^x = 23$

b) $y^{x-1} = 8$

c) $x^8 = u$

d) $(x-3)^2 = 6$

More about the relationship between $f(x) = b^x$ and $g(x) = \log_b x$:

Because $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of one another, $f(g(x)) = x$ and $g(f(x)) = x$.

This gives us…

$$\boxed{\begin{array}{c} \log_b b^x = x \\[1em] b^{\log_b x} = x \end{array}}$$

Example 14: Simplify $\log_3 3^{x+5}$.

Example 15: Simplify $\log_2 2^{12}$.

Example 16: Simplify $\ln e^{-2}$.

Example 17: Simplify $5^{\log_5 y}$.

Example 18: Simplify $3^{\log_3 \sqrt{2}}$.

Example 19: Simplify $e^{\ln(x^2+1)}$.

The common logarithm:

Often $\log$ is used to mean $\log_{10}$. The logarithm of base 10 is called the *common logarithm*.

Example 20: Evaluate $\log \sqrt[3]{10}$.

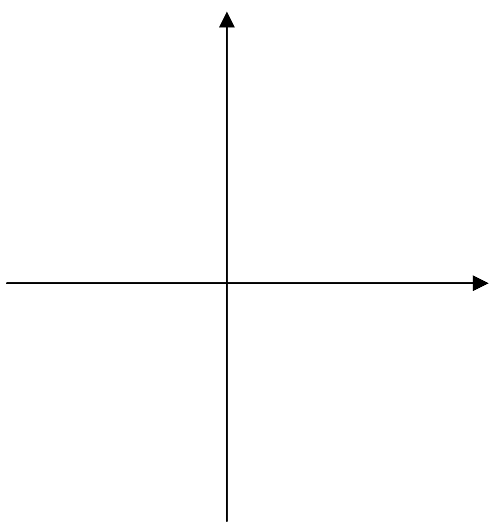
Evaluating logs on your calculator:

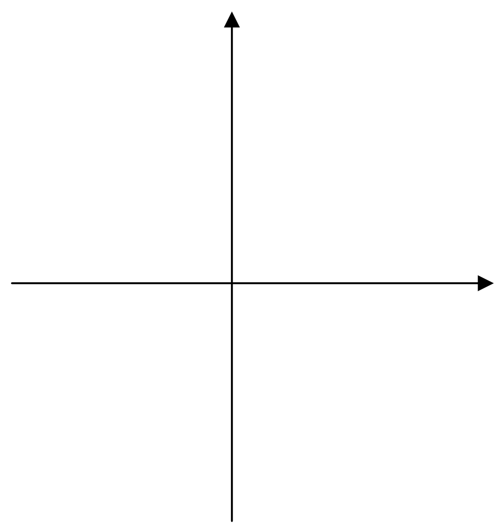
<u>Example 21:</u> Evaluate $\log_{10} 72$ on your calculator.

<u>Example 22:</u> Evaluate $\ln 12$ on your calculator.

Graphs of logarithmic functions:

To get the graph of $f(x) = \log_b x$, start with the graph of $f(x) = b^x$ and reflect it about the line $y = x$.
Why?

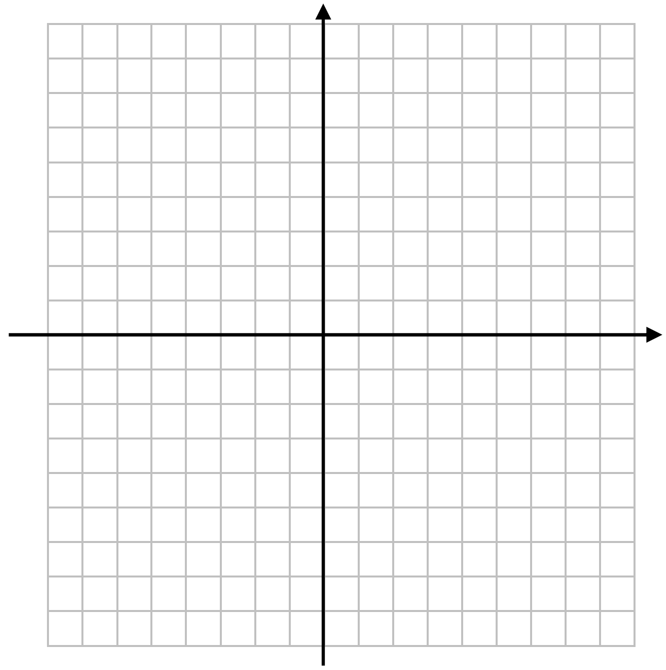




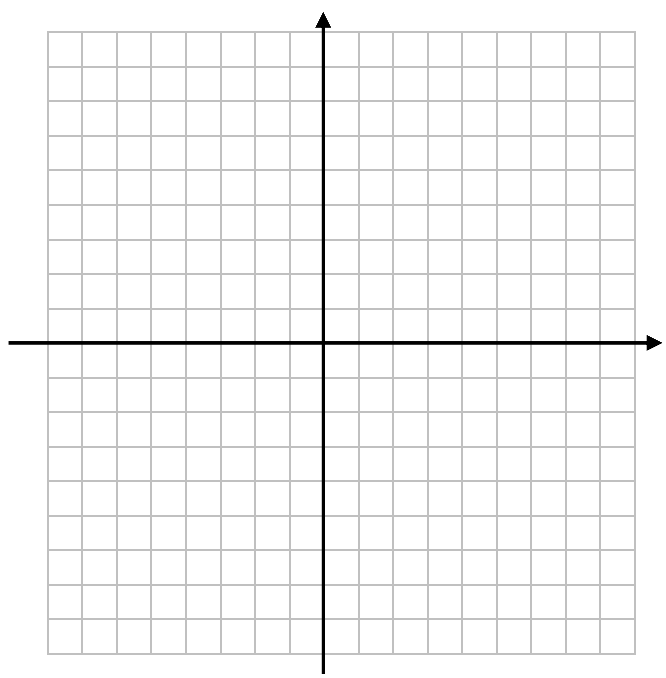
Facts about the graphs of $y = b^x$ and $y = \log_b x$:

$y = b^x$	$y = \log_b x$
Domain:	Domain:
Range:	Range:
Asymptote:	Asymptote:
Passes through:	Passes through:

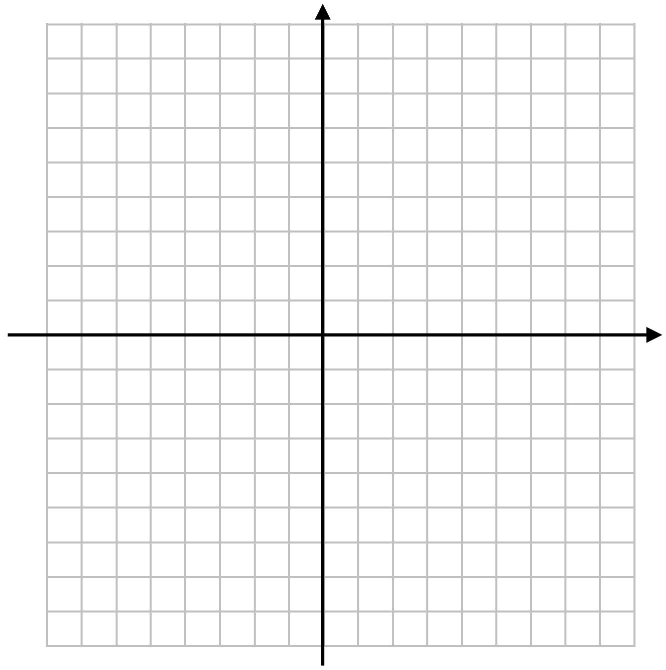
Example 23: Sketch the graph of $f(x) = \log_2 x$ by plotting points.



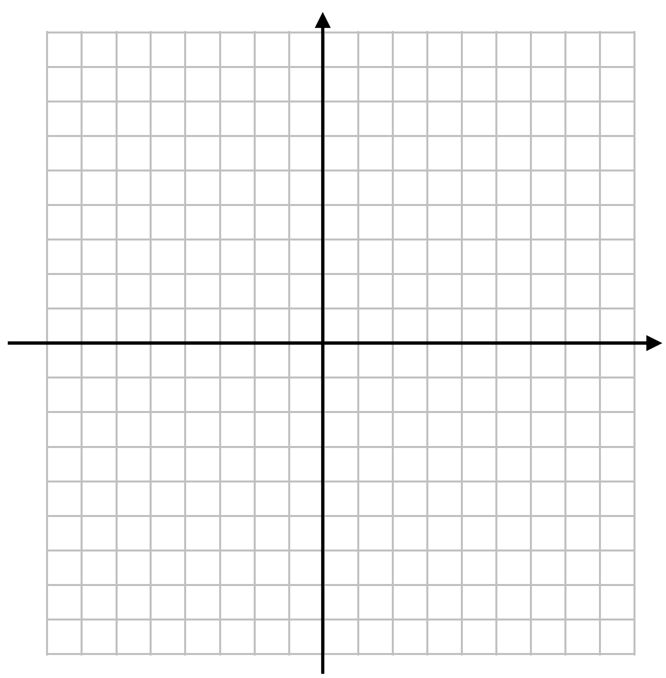
Example 24: Graph $f(x) = \log_2(x+3)$.



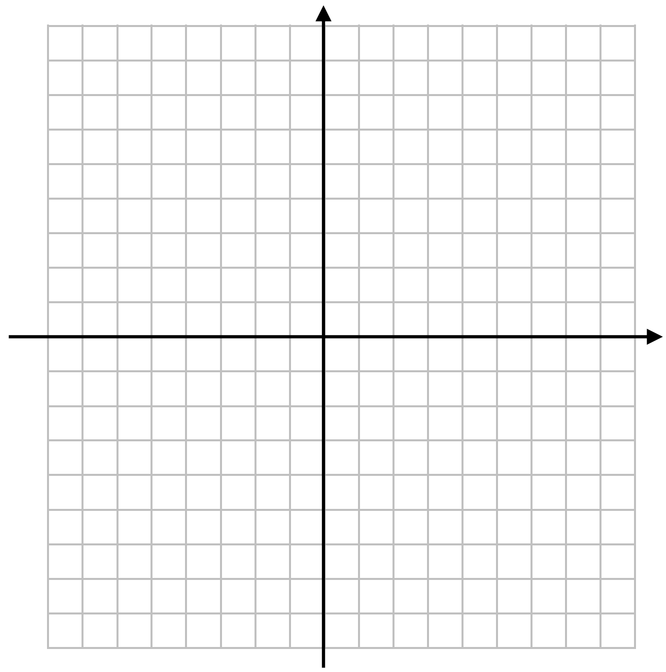
Example 25: Graph $y = -\ln x$.



Example 26: Graph $g(x) = \ln(x+2) - 1$.



Example 27: Graph $f(x) = -2 - \log_2(x-3)$.



Important: When graphing logarithmic and exponential functions, ALWAYS label the reference point with its coordinates. Also label the asymptote.

Example 28: Find the function of the form $y = \log_a x$ whose graph includes the point $(64, 3)$.

Finding the domain of logarithmic functions:

Example 29: Find the domain of $f(x) = \log_3(x-4)$.

<u>Example 30:</u> Find the domain of $f(x) = \log_5(x^2)$.

<u>Example 31:</u> Find the domain of $f(x) = \ln(x^2 + 6)$.

<u>Example 32:</u> Find the domain of $g(x) = \ln(3 - 2x)$.

<u>Example 33:</u> Find the domain of $h(x) = \ln(-x)$.

Solving simple logarithmic equations:

<u>Example 34:</u> Solve for *x*.

$$\log_2(x-1) = 5$$

<u>Example 35:</u> Solve for *x*.

$$\log_x 7 = \frac{1}{2}$$