6.1: Matrix Solutions to Linear Systems

Earlier we discussed using the substitution and elimination methods of solving a system of linear equations in two variables. However, while these techniques also work for larger systems, they can be lengthy and cumbersome. Now, we will look at techniques to streamline the process for any size system.

Matrices:

A *matrix* is a rectangular array of numbers written within brackets.

Examples: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 9 & 12 \\ -2 & 4 & 8 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$, $E = \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$, $F = \begin{bmatrix} 7 \end{bmatrix}$

Each number is an *element*. The *size* is expressed as $m \times n$, where *m* is the number of rows and *n* is the number of columns.

Special matrices:

- *Row Matrix:* one row (matrices *D* and *F* in the example above).
- *Column Matrix:* one column (matrices *E* and *F* in the example above).
- *Square Matrix:* same number of rows as columns (matrices *C* and *F* in the example above).

The *position* of an element is given by the row and column in the matrix where the element is.

Example 1: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & 0 & -8 \\ 7 & -3 & 9 \end{bmatrix}$. Element a_{ij} is in row *i* and column *j*.

The *principal diagonal* is the collection of elements a_{ii}. (elements with row number and column number the same).

Example 2: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & 0 & -8 \\ 7 & -3 & 9 \end{bmatrix}$.

Augmented Matrices:

In the solution of systems of linear equations, we will use *augmented matrices*.

<u>Example 3:</u> $\begin{cases} 2x_1 + x_2 = 7 \\ 3x_1 + 2x_2 = 12 \end{cases}$

Each equation gives us a row in the augmented matrix. The coefficients and the constants are the elements.

Augmented matrix: $\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 3 & 2 & 12 \end{array}\right]$

Each column to the left of the vertical bar represents the coefficients of a particular variable.
The column to the right represents the "constants" in the equations. These constants should be written on the right-hand side of the equals sign.

<u>Example 4:</u> $\begin{array}{l} 2x - 3y + z = 9 \\ -x + 8z = -2 \\ 4x - 7y + 6z = 0 \end{array}$. Write the augmented matrix for the system.

<u>Example 5:</u> Give the system represented by $\left[\begin{array}{ccc|c} -8 & 0 & 7 & 1 \\ 5 & 3 & -6 & 2 \\ 0 & 2 & -1 & -3 \end{array}\right]$.

Example 6: Solve the system represented by this matrix. $\left[\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$.

This is called *back-substitution*.

A matrix with 1's down the principal diagonal and 0's below the diagonal is said to be in *row-echelon form*. We can put any matrix in row-echelon form using row operations to produce equivalent matrices.

Two augmented matrices are *row-equivalent* if they correspond to equivalent linear systems (i.e. they have the same solution set). This is denoted by the symbol □ (e.g. $A \square B$).

To solve a system using augmented matrices, we use the following *row operations* to produce new row-equivalent matrices.

Operations that produce row-equivalent matrices:

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following:

a. Two rows are interchanged $\left(R_i \leftrightarrow R_j\right)$.
b. A row is multiplied by a nonzero constant $\left(kR_i \to R_i\right)$.
c. A constant multiple of one row is added to another row $\left(kR_j + R_i \to R_i\right)$.

Note: The arrow $\to$ means "replaces".

Gauss-Jordan Elimination with Back-Substitution:

We use the row operations to write the augmented matrix in row-echelon form. Then use back-substitution to solve the system. Don't forget to check your answers!

Example 7: Solve the system using Gaussian elimination and back-substitution.

$$\begin{aligned} x-2y-z&=2 \\ 2x-y+z&=4 \\ -x+y-2z&=-4 \end{aligned}$$

Example 8: Solve the system using Gaussian elimination and back-substitution.

$$\begin{aligned} 3a+b-c&=0 \\ 2a+3b-5c&=1 \\ a-2b+3c&=-4 \end{aligned}$$

Gauss-Jordan Elimination:

A matrix with 1's down the main diagonal and 0's above and below them is said to be in *reduced-row-echelon form*.

Examples:

Example 9: $\begin{cases} 2x_1 + x_2 = 7 \\ 3x_1 + 2x_2 = 12 \end{cases}$

Our goal: to use these row operations to derive an augmented matrix of the form $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$.

To do this, start in the upper left and work *counterclockwise*.

Example 10: Solve the system using Gauss-Jordan elimination.

$$\begin{aligned} 3y - z &= -1 \\ x + 5y - z &= -4 \\ -3x + 6y + 2z &= 11 \end{aligned}$$

Example 11: Solve the system using Gauss-Jordan elimination.

$$\begin{aligned} 2x + y &= z + 1 \\ 2x &= 1 + 3y - z \\ -x + y + z &= 4 \end{aligned}$$