1.4: Using the Definitions of the trig functions

$(-,+)$

$\cos\theta < 0$
$\sin\theta > 0$
$\tan\theta < 0$

$(+,+)$

$\cos\theta > 0$
$\sin\theta > 0$
$\tan\theta > 0$

$\cos\theta < 0$
$\sin\theta < 0$
$\tan\theta > 0$

$(-,-)$

$\cos\theta > 0$
$\sin\theta < 0$
$\tan\theta < 0$

$(+,-)$

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

$$\tan\theta = \frac{y}{x}$$

Identities: An identity is a mathematical statement of equality that is true for all values of the variable.

Reciprocal Identities:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Quotient Identities

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

Why? $\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{\cancel{r}} \cdot \frac{\cancel{r}}{x}$

$= \frac{y}{x} = \tan\theta$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta$$

Pythagorean Identities:

$\sin^2\theta + \cos^2\theta = 1$ ← ★ ★ Know this!!

Note: $\sin^2\theta$ means $(\sin\theta)^2$
$\cos^2\theta$ means $(\cos\theta)^2$
(shorthand)

Why is this identity true?

From Pythagorean Theorem, $x^2 + y^2 = r^2$

Divide both sides by r^2: $\frac{x^2+y^2}{r^2} = \frac{r^2}{r^2}$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

Also true for angles on the axes (quadrantal angles)

2 more Pythagorean Identities

★ ★

$$\boxed{\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}}$$

★ ★

Can use the 1st one to get the other two:

$$\sin^2\theta + \cos^2\theta = 1$$

Divide both sides by $\cos^2\theta$:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + \left(\frac{\cos\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2$$

$$(\tan\theta)^2 + (1)^2 = (\sec\theta)^2$$

$$\tan^2\theta + 1 = \sec^2\theta$$

Start over with $\cos^2\theta + \sin^2\theta = 1$

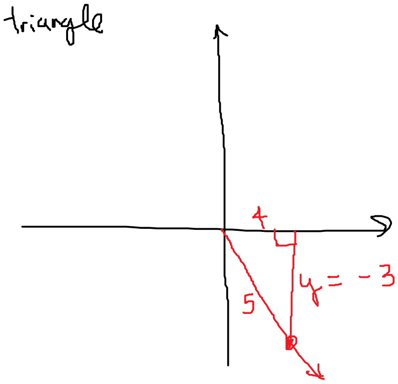
Divide both sides by $\sin^2\theta$:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Example: Find $\sin\theta$, given that $\cos\theta = \frac{4}{5}$ and θ is in Quadrant IV.

Method 1: Draw a triangle



$\cos\theta = \frac{4}{5} = \frac{x}{r}$

can use $x = 4$, $r = 5$

Find y: $4^2 + y^2 = 5^2$

$16 + y^2 = 25$

$y^2 = 9$

$y = \pm\sqrt{9}$

$y = \pm 3$

Use the quadrant to decide + or −

$y = -3$ (Quadrant IV)

$\sin\theta = \frac{y}{r} = \frac{-3}{5}$

$\boxed{\sin\theta = -\frac{3}{5}}$

Method 2: Use the Pythagorean Identities

$\cos\theta = \frac{4}{5}$, Q IV (I often write Q4)

$\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{4}{5}\right)^2 + \sin^2\theta = 1$

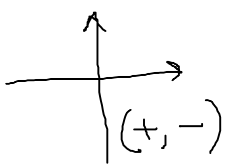
$\frac{16}{25} + \sin^2\theta = 1$

$\sin^2\theta = 1 - \frac{16}{25}$

$\sin^2\theta = \frac{25}{25} - \frac{16}{25}$

$\sin^2\theta = \frac{9}{25}$

$\sin\theta = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$



Quadrant 4, so choose negative $\boxed{\sin\theta = -\frac{3}{5}}$

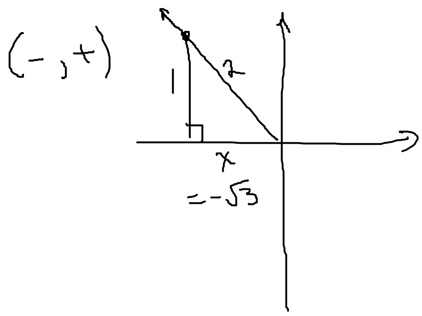
What if it had asked us to find $\csc\theta$?

1st find $\sin\theta = -\frac{3}{5}$. then $\boxed{\csc\theta = -\frac{5}{3}}$

($\csc\theta$ is reciprocal of $\sin\theta$)

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{\cancel{5}} \cdot \frac{\cancel{5}}{4} = \boxed{-\frac{3}{4}}$

Ex: Find $\tan\theta$, given that θ is in Quadrant II and $\sin\theta = \frac{1}{2}$



$\sin\theta = \frac{1}{2} = \frac{y}{r}$

use $y = 1$
$r = 2$

$1^2 + x^2 = 2^2$

$1 + x^2 = 4$

$x^2 = 3$

$x = \pm\sqrt{3}$

Quadrant 2, so choose negative

$x = -\sqrt{3}$

$\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}}$

$\boxed{\tan\theta = -\frac{1}{\sqrt{3}}}$ ⟵ exact value of $\tan\theta$

Note: $\tan\theta \approx 0.5774$ ⟵ approximate value of $\tan\theta$

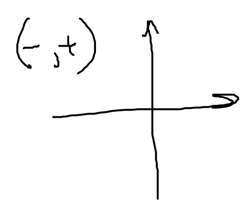
If we rationalize denominator, we would get

$\boxed{\tan\theta = -\frac{\sqrt{3}}{3}}$

Ex: Suppose $\sin\theta = \frac{\sqrt{2}}{6}$ and $\cos\theta < 0$. Find the other trig function values.

What quadrant is this in?

$\left.\begin{array}{l} \sin\theta > 0 \Rightarrow y \text{ is positive} \\ \cos\theta < 0 \Rightarrow x \text{ is negative} \end{array}\right\} \Rightarrow$ Quadrant 2



then similar to what we did in other examples

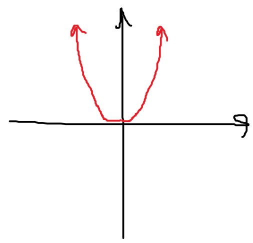
Range of the trig functions

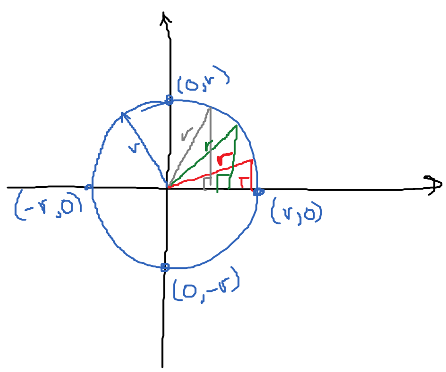
Range: the set of all possible output values for a functions.

Note: Consider $f(x) = x^2$

set of inputs $\longrightarrow$ Domain: $(-\infty, \infty)$

set of outputs $\longrightarrow$ Range: $[0, \infty)$





What is largest possible value for $\sin\theta = \frac{y}{r}$?

Notice: x cannot be larger than r

y cannot be large than r

So max of $\sin\theta$ is $\frac{r}{r} = 1$

max of $\cos\theta$ is 1

if y is negative, can get to $\sin\theta = -1$ on y-axis.

$$-1 \leq \sin\theta \leq 1$$

similarly $\quad -1 \leq \cos\theta \leq 1$

Trig functions	Range
$\sin\theta$, $\cos\theta$	$[-1, 1]$
$\sec\theta$, $\csc\theta$	$(-\infty, 1] \cup [1, \infty)$
$\tan\theta$, $\cot\theta$	$(-\infty, \infty)$

Are these possible?

Ex: $\sin\theta = 2.5$

Not possible

Ex: $\sin\theta = 0.25$

Possible

Ex: $\sec\theta = \frac{2}{3}$

Not possible