7.1: Oblique Triangles + the Law of Sines
7.2: The Ambiguous Case of the Law of Sines
7.3: The Law of Cosines:

Oblique Triangle: non-right triangle (does not have a 90° angle)

Deciding which Law to Use:

A = angle, S = side

Given info	Use:
SAA, AAS, ASA	Law of Sines
SSA	Law of Sines (ambiguous case, could have 1 triangle, 2 triangles, or no triangles)
SSS, SAS	Law of Cosines

To solve a triangle means to find all the missing sides and angles.

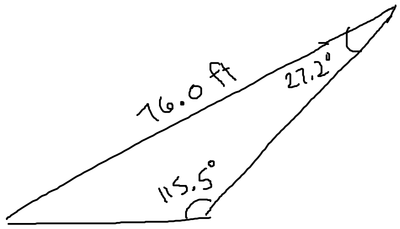
Law of Sines

In any triangle with sides a, b, c,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

(a across from angle A, b across from Angle B, etc)

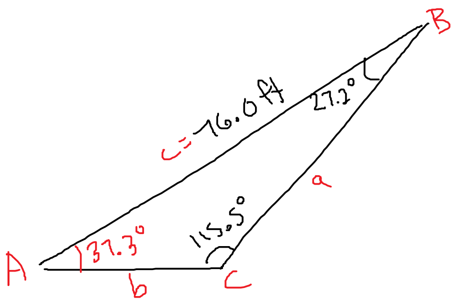
Ex: Find the remaining sides and angles.



SAA

(or AAS)

Use Law of Sines



$$A = 180° - 115.5° - 27.2° = 37.3°$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{76.0\text{ ft}}{\sin(115.5°)} = \frac{b}{\sin(27.2°)}$$

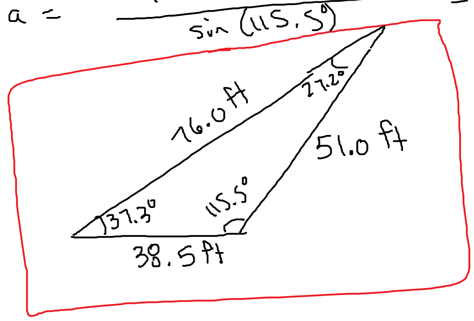
$$\sin 27.2° \left(\frac{76.0\text{ ft}}{\sin 115.5°} \right) = b$$

$$b = \frac{76\text{ ft} \sin 27.2°}{\sin 115.5°} = 38.489\text{ ft} \approx 38.5\text{ ft}$$

Find 3rd side: $\frac{a}{\sin A} = \frac{c}{\sin C}$

$$\frac{a}{\sin(37.3°)} = \frac{76.0\text{ ft}}{\sin(115.5°)}$$

$$a = \frac{76.0\text{ ft} \cdot \sin(37.3°)}{\sin(115.5°)} = 51.026\text{ ft} \approx 51.0\text{ ft}$$



7.3: Law of Cosines

Law of Cosines

In any triangle ABC, with side a across from A, side b across from B, side c across from C,

these three identities are true:

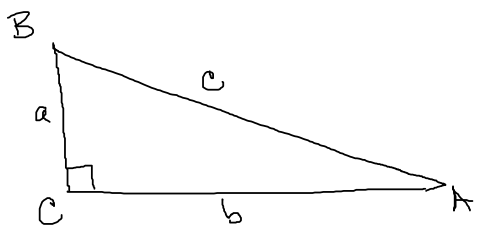
$$\boxed{c^2 = a^2 + b^2 - 2ab\cos(C)}$$

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

The Law of Cosines is also ~~This~~ called the **Generalized Pythagorean Theorem**

Ex: Suppose we have a right triangle.



Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

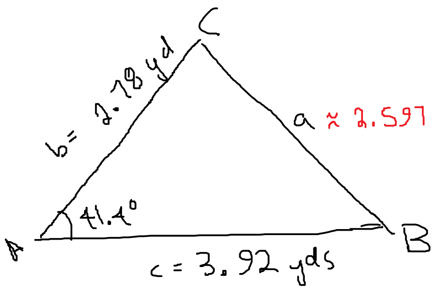
$$c^2 = a^2 + b^2 - 2ab\cos(90^\circ)$$

$$c^2 = a^2 + b^2 - 2ab(0)$$

$$c^2 = a^2 + b^2 \quad \text{Familiar?}$$

Ex: Solve the triangle.

$A = 41.4^\circ$, $b = 2.78$ yds, $c = 3.92$ yds



SAS $\Rightarrow$ use Law of Cosines

$$(a)^2 = (2.78\text{yd})^2 + (3.92\text{yd})^2 - 2(2.78\text{yd})(3.92\text{yd})\cos(41.4^\circ)$$

$$a^2 = 6.74598$$

$$a = \pm\sqrt{6.74598}$$

$$a \approx \pm 2.597$$

side of a triangle, so choose $a = +2.597$

Use Law of Sines to find the smaller of the remaining two angles.

choosing the smaller angle to solve for guarantees it won't be obtuse.

$$\frac{(2.78 \text{ yds})}{\sin(B)} = \frac{(2.597 \text{ yds})}{\sin(41.4^\circ)}$$

$$2.78\text{yds} \sin(41.4^\circ) = 2.597 \text{ yds} \sin(B)$$

$$\frac{2.78 \cancel{\text{yds}} \sin(41.4^\circ)}{2.597 \cancel{\text{yds}}} = \sin(B)$$

$$0.70791 \approx \sin(B)$$

ref angle for B: $\hat{B} = \sin^{-1}(0.70791)$

$$\approx 45.065 \approx 45.1^\circ$$

(can't be obtuse, so we know angle B is acute (Quadrant I).

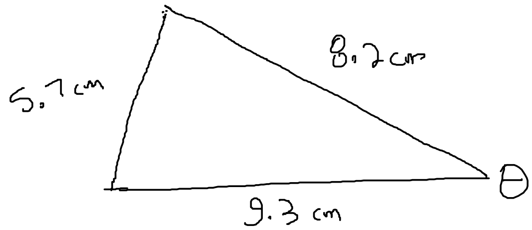
So $B = 45.1^\circ$

Find 3rd angle: $180^\circ - 45.1^\circ - 41.4^\circ = 93.5^\circ$

Missing triangle parts are:

$$\boxed{\begin{array}{l} B = 45.1^\circ \\ C = 93.5^\circ \\ a = 2.60 \text{ yds} \end{array}}$$

Ex: A triangle has sides 9.3 cm, 5.7 cm, 8.2 cm. Find the angles.



SSS $\Rightarrow$ Use Law of Cosines

$$(5.7\text{cm})^2 = (8.2\text{cm})^2 + (9.3\text{cm})^2 - 2(8.2\text{ cm})(9.3\text{ cm})\cos(\theta)$$

$$32.49\text{ cm}^2 = 67.24\,\text{cm}^2 + 86.49\,\text{cm}^2 - 152.52\text{cm}^2\cos(\theta)$$

$$-121.27\text{ cm}^2 = -152.52\text{ cm}^2\cos(\theta)$$

$$\frac{-121.27\,\cancel{\text{cm}^2}}{-152.52\,\cancel{\text{cm}^2}} = \cos(\theta)$$

$$0.7951 \approx \cos(\theta)$$

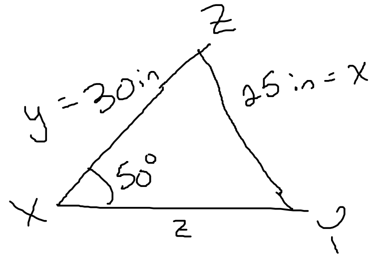
$$\theta \approx \cos^{-1}(0.7951) \approx 37.33^\circ$$

use law of sines to find the other angles.

7.2: The Ambiguous Case of the Law of Sines

The SSA case is called the ambiguous case because it can result in 1 triangle, 2 triangles, or no triangle.

Ex: Suppose Triangle XYZ has $X = 50^\circ$, $x = 25\text{in}$, $y = 30\text{in}$. Solve the triangle. (x across from angle X)



SSA
Law of Sines

$$\frac{(25\text{ in})}{\sin(50^\circ)} = \frac{(30\text{ in})}{\sin(Y)}$$

$$\frac{\sin(50^\circ)}{25\text{ in}} = \frac{\sin(Y)}{30\text{ in}}$$

$$\frac{30\cancel{\text{in}}}{1}\left(\frac{\sin 50^\circ}{25\cancel{\text{in}}}\right) = \sin(Y)$$

$$0.91925 = \sin(Y)$$

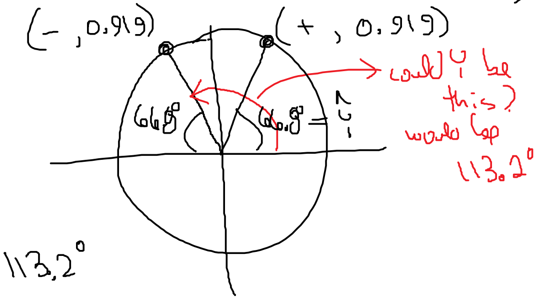
$$\sin^{-1}(0.91925) = \hat{Y} \text{ (ref angle for } Y)$$

$$\hat{Y} \approx 66.817^\circ \Rightarrow 66.8^\circ$$

Is there a second triangle?

Can there be a QII angle with $\sin Y = 0.91975$

Y could be in QI or QII (sin Y is positive)



If there is, it would be:
supplement of 66.8°: $180^\circ - 66.8^\circ = 113.2^\circ$

$50^\circ + 113.2^\circ = 163.2^\circ < 180^\circ \Rightarrow 2^{nd}$ triangle, with $Y = 113.2^\circ$

If we added these and got more than 180°, then it is impossible to make a QII triangle

Here we have 2 triangles

Triangle #1	Triangle #2
$Y = 66.8^\circ$	$Y = 113.2^\circ$
$Z = 180^\circ - 66.8^\circ - 50^\circ = 63.2^\circ$	$Z = 180^\circ - 113.2^\circ - 50^\circ = 16.8^\circ$
$\frac{z}{\sin 63.2^\circ} = \frac{25}{\sin 50^\circ}$	$\frac{z}{\sin 16.8^\circ} = \frac{25\sin}{\sin 50^\circ}$