

1342-Notes_Navidi_7-3_sampling-distributions-Central-Limit-Theorem

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7.3.1

7.3: Sampling Distributions and the Central Limit Theorem

Suppose we want to estimate some characteristic of a population; also suppose that it is unrealistic to survey or measure the entire population. Therefore, we select a sample from that population, and we survey or measure the characteristic of interest. If we record the values of the characteristic for the sample, we can compute a sample statistic: the mean, standard deviation, proportion, etc.

Recall: A *parameter* is a numerical summary of a population; a *statistic* is a numerical summary of a sample. (For example, the population mean and population standard deviation are parameters; the sample mean and sample standard deviation are statistics.)

The *sampling distribution of a statistic* is the probability distribution of all possible values for that statistic computed from all possible samples of fixed size n .

The *sampling distribution of the sample mean* is the probability distribution of all possible values for that the sample mean $\bar{x}$ computed from all possible samples of fixed size n .

Example 1: Consider the following population data.

- Calculate the population mean.
- Calculate the means for all possible samples of size 1.
- Calculate the means for all possible samples of size 2.
- Calculate the means for all possible samples of size 3.
- Calculate the means for all possible samples of size 4.
- Calculate the means for all possible samples of size 5.
- Summarize the sample means in a series of dotplots (one dotplot for each sample size).
- For each sample size, calculate the probability that the sample mean is within 5 units of the population mean.

X
50
57
54
45
50

$$\mu = \frac{50 + 57 + 54 + 45 + 50}{5} = 51.2$$

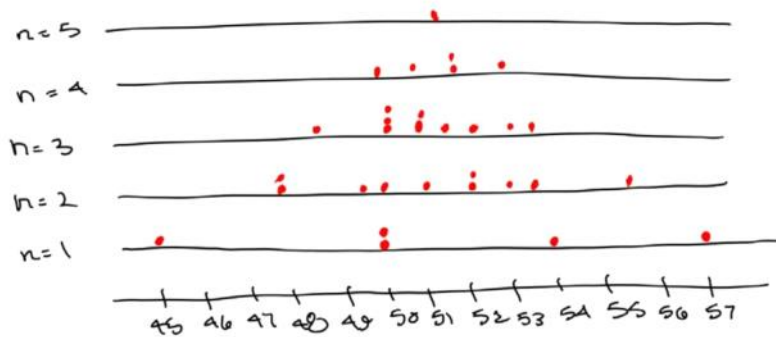
$$n=2: \begin{array}{ll} \{50, 57\} & \bar{x} = 53.5 \\ \{50, 54\} & \bar{x} = 52 \\ \{50, 45\} & \bar{x} = 47.5 \\ \{57, 54\} & \bar{x} = 50 \\ \{57, 45\} & \bar{x} = 51 \\ \{54, 45\} & \bar{x} = 49.5 \end{array}$$

b) $n=2$		c) $n=3$	
Sample	$\bar{x}$		$\bar{x}$
50, 51	50.5	54, 45, 50	49.67
50, 51	50.5	57, 45, 50	50.67
50, 45	47.5	55, 54, 50	49.67
50, 50	50	51, 54, 45	52
57, 54	55.5	50, 45, 50	48.33
57, 45	51	50, 54, 50	51.33
57, 50	53.5	54, 45, 50	49.67
54, 45	49.5	50, 51, 50	50.33
54, 50	52	50, 57, 45	50.67
45, 50	47.5	50, 57, 54	53.67

d) $n=4$	$\bar{x}$
57, 54, 45, 50	51.5
50, 54, 45, 50	49.75
50, 57, 45, 50	50.5
50, 57, 54, 50	52.75
50, 57, 54, 45	51.5

$$a) \mu = \frac{50 + 57 + 54 + 45 + 50}{5} = 51.2$$

$$b) \bar{x} = 50, 57, 54, 45, 50$$



Simulations:

Rice Virtual Lab in Statistics: <http://onlinestatbook.com/rvls/index.html>
 (A public domain resource created by Dr. David Lane of Rice University)
http://onlinestatbook.com/stat_sim/sampling_dist/index.html

<http://www.rossmanchance.com/>

(Created by Dr. Allen Rossman and Dr. Beth Chance of California Polytechnic State University – San Luis Obispo; used with their permission.)

<http://www.rossmanchance.com/applets/OneSample.html>

These examples illustrate the Central Limit Theorem, an extremely important result. Much of the usefulness of statistics depends on this theorem.

The Central Limit Theorem:The Central Limit Theorem

For a large population and sufficiently large sample sizes, the sample mean $\bar{x}$ is approximately normally distributed.

In other words: For a large population and sufficiently large sample sizes, the sampling distribution of the mean of a random variable is approximately normal.

This is true regardless of the shape of the underlying distribution of the random variable (i.e., regardless of whether the variable is normally distributed in the population from which the samples are drawn).

What is a “sufficiently large sample size” for the sampling distribution of the sample mean to be normal?

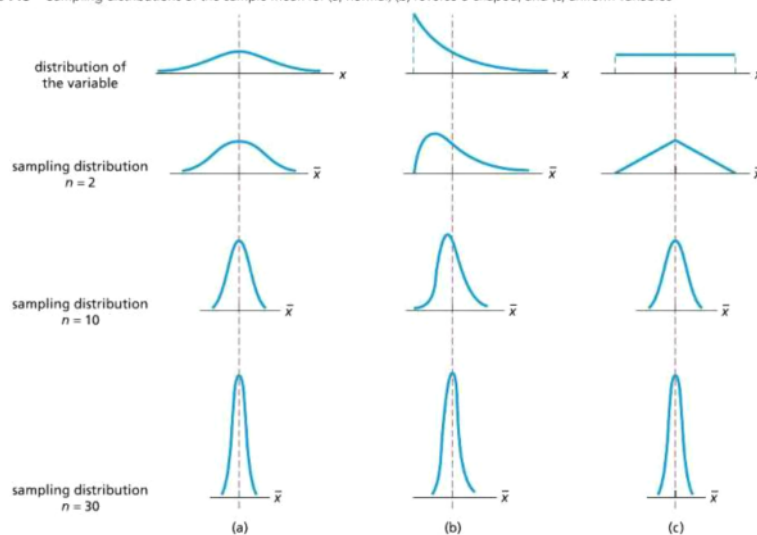
For a variable that is normally distributed in the underlying population: The sample mean $\bar{x}$ is normally distributed, regardless of sample size.

For a variable that is NOT normally distributed in the underlying population: As the sample size increases, the sampling distribution of the sample mean will become closer and closer to a normal distribution (i.e., it will become more bell-shaped). The less closely the underlying distribution (in the population) resembles a normal distribution, the larger the sample size requirement.

Rule of Thumb: According to a frequently-used (and conservative) rule of thumb, a sample size of at least 30 is sufficient to assume that the Central Limit Theorem is true, if the variable is not normally distributed in the population.

sample means are normally distributed.

FIGURE 7.6 Sampling distributions of the sample mean for (a) normal, (b) reverse-J-shaped, and (c) uniform variables



Mean and standard deviation of the sampling distribution of the sample means:

Mean and Standard Error of the Sampling Distribution:

Suppose a random variable X has population mean μ and population standard deviation σ .

Then, for samples of size n , the sampling distribution of the sample means will have:

Mean: $\mu_{\bar{x}} = \mu$ $\mu_{\bar{x}}$ is the mean of all possible samples taken from that population, with a specific sample size

Standard deviation: $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ $\sigma_{\bar{x}}$ is the SD of all possible samples of that size from that population

The standard deviation of the sample mean $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ is called the *standard error of the mean*.

Note: The variance in the population will be n times as large as the variance of the sampling

distribution of the sample means (i.e., $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$ is equivalent to $n\sigma_{\bar{x}}^2 = \sigma^2$).

Calculating the probability of a sample's results:

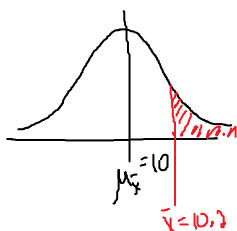
Example 2: Suppose a coffee maker dispenses servings of coffee that are normally distributed with a mean size of 10 ounces and a standard deviation of 0.3 ounces. Find the probability that:

- A sample of 8 servings of coffee will result in a mean greater than 10.2 ounces.
- A sample of 8 servings of coffee will result in a mean between 9.8 and 10.2 ounces.
- A sample of 8 servings of coffee will result in a mean below 9.8 ounces.
- An individual serving of coffee is less than 9.8 ounces.
- A sample of 12 servings of coffee will result in a mean below 9.8 ounces.
- The sampling error made in using a sample of 8 coffee servings to estimate the size of all coffee servings will be at most 0.1 ounce.

a) Sample size $n=8$ has $\bar{x} > 10.2$

$$\mu = 10$$

$$\sigma = 0.3$$



$$\mu_{\bar{x}} = \mu = 10$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0.3}{\sqrt{8}} \approx 0.106066 \quad (\text{store in calculator})$$

Find z-score for $\bar{x} = 10.2$

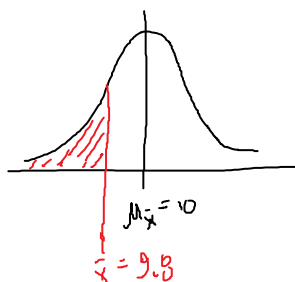
$$Z = \frac{x - \mu}{\sigma} \text{ for an individual data point}$$

$$Z = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{10.2 - 10}{\left(\frac{0.3}{\sqrt{8}}\right)} \approx 1.895618083 \approx 1.89$$

Look up $z = 1.89$ in normal table $\Rightarrow P(Z < 1.89) = 0.9706$

$$P(\bar{x} > 10.2) = P(Z > 1.89) = 1 - 0.9706 = \boxed{0.0294}$$

c) A sample size $n=8$ has $\bar{x} < 9.8$

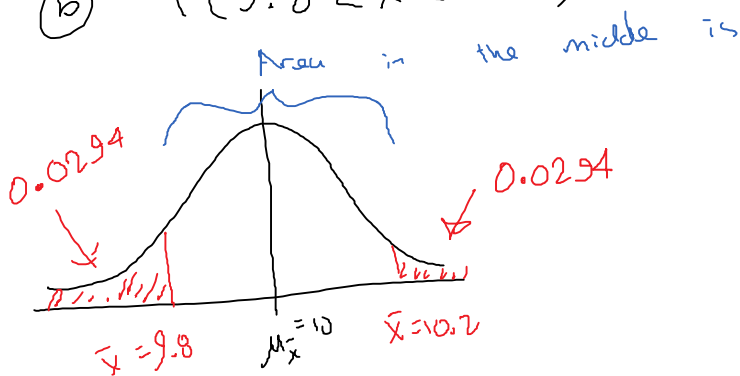


$$Z\text{-score for } \bar{x} = 9.8 \text{ is } Z = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{9.8 - 10}{\left(\frac{0.3}{\sqrt{8}}\right)} \approx -1.89$$

From table, $P(Z < -1.89) = 0.0294$

$$\Rightarrow P(\bar{x} < 9.8) = \boxed{0.0294}$$

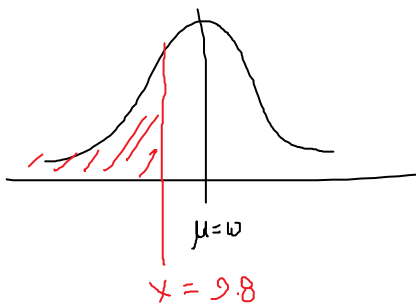
b) $P(9.8 < \bar{X} < 10.2)$



$$1 - 2(0.0294) = 1 - 0.0588 = \boxed{0.9412}$$

Area in tails = $2(0.0294) = 0.0588$

c) Prob that an individual serving is less than 9.8 oz



$$Z = \frac{x - \mu}{\sigma} = \frac{9.8 - 10}{0.3} = -0.67$$

From table, $P(Z < -0.67) = \boxed{0.2514}$

e) Prob that for $n = 12$, $\bar{X} < 9.8$

$$\mu_{\bar{x}} = \mu = 10.0$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0.3}{\sqrt{12}} \approx 0.0866$$

Z-score is $Z = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{9.8 - 10}{\left(\frac{0.3}{\sqrt{12}}\right)} \approx -2.31$

Look up $Z = -2.31$ in table

$$P(\bar{x} < 9.8) = P(Z < -2.31) = \boxed{0.0104}$$

note smaller than for $n=8$

Example 3: Suppose that a pizza company's records showed that the mean delivery time is 19.3 minutes and that the standard deviation of the delivery times is 6.8 minutes.

- a) What sample size is required for us to be able to calculate probabilities using the normal model?
- b) What is the probability that a sample of 35 orders has a mean delivery time between 20 and 23 minutes?
- c) What is the probability that a sample of 35 orders has a mean delivery time over 26 minutes?
- d) Calculate the value for which there is a 0.05 probability that the mean delivery time is below that value (for a sample of 35).