Notes Matrices

- A ***matrix*** is a rectangular array of numbers arranged in rows and columns placed inside brackets.
    - The numbers inside the brackets of a matrix are called elements.

Example of a Matrix: $\begin{bmatrix} 13 & 15 & 21 \\ 7 & 5 & 6 \\ 12 & 1 & 41 \end{bmatrix}$

We will be using matrices to solve Systems of Linear Equations.

- An ***Augmented Matrix*** is a matrix that is used to represent a system of Linear Equations. It has a vertical bar separating the columns of the matrix into 2 groups (one for the coefficients of the variables in the linear system and the other for the answers).
    - Note: If a variable is missing, we assign it the coefficient of ***ZERO***.

Rewrite the following system of equations in an augmented matrix

System of Equation	Augmented Matrix
Ex1: $\begin{array}{l} 3x+y-z=9 \\ -x-y+3z=3 \\ x+2y-z=0 \end{array}$	
Ex2: $\begin{array}{r} x-3y+2z=-12 \\ y+4z=-3 \\ z=2 \end{array}$	
Ex3: $\begin{array}{l} 3x-5z=-12 \\ 4y-z=5 \\ 2x-3y=-4 \end{array}$	

- One method to solve a system of linear equations using a matrix is to get the matrix in **Echelon Form**, which means to have only 1s in your main diagonal (going from upper-left to lower-right) and 0s below the ones.
- Once you are in Echelon Form, you will use back substitution to solve your system.

Examples of matrices in Echelon Form: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Matrices do not always come in echelon form; we will use row operations to get in this format.

Row Operations

There are 3 row operations that produce matrices that represent systems with the same solution set.

Row Operation	Notation	Example
Interchange 2 Rows	$R_1 \leftrightarrow R_3$	$\left[\begin{array}{ccc\|c} -1 & -1 & 3 & 3 \\ 3 & 18 & -12 & 21 \\ 1 & 2 & -1 & 0 \end{array}\right]$
Multiply a Row by a non-zero number	$2R_2 \rightarrow R_2$	$\left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 3 & 18 & -12 & 21 \\ -1 & -1 & 3 & 3 \end{array}\right]$
Multiply a Row by a non-zero number and then add the product to any other row	$\frac{1}{6}R_2 + R_3 \rightarrow R_3$	$\left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 6 & 36 & -24 & 42 \\ -1 & -1 & 3 & 3 \end{array}\right]$

To solve a system of linear equations using matrices we would

1. Write system as an augmented matrix
2. Use row operations to get row equivalent matrix in Row Echelon Form
3. Use Substitution to solve for the variables. Answer solution in an ordered triple.

For Example

System of equations	$\begin{aligned} 3x + y - z &= 9 \\ -x - y + 3z &= 3 \\ x + 2y - z &= 0 \end{aligned}$
Rewrite system of equations as a augmented matrix	$\left[\begin{array}{ccc\|c} 3 & 1 & -1 & 9 \\ -1 & -1 & 3 & 3 \\ 1 & 2 & -1 & 0 \end{array}\right]$
Use row operations to change matrix to a row equivalent matrix in row echelon form.	$\cdots\left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$
Rewrite matrix from row echelon form to system of equations.	$\begin{aligned} x + 2y - z &= 0 \\ y + 2z &= 3 \\ z &= 2 \end{aligned}$
Solve the system using back substitution	$(4, -1, 2)$

Solve the following System of Equations

Ex4: $\begin{aligned} 3x + y - z &= 9 \\ -x - y + 3z &= 3 \\ x + 2y - z &= 0 \end{aligned}$	Ex5: $\begin{aligned} 2x + y - z &= 10 \\ -x + 2y + 2z &= 6 \\ x - 3y + 2z &= -15 \end{aligned}$

Special Cases

Just as in system of equations in 2 variables, you may encounter systems with no solution or infinitely many solutions.

Matrix is Row Echelon Form	System of Equation	How many solutions	Solution
Ex6: $\left[\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$			
Ex7: $\left[\begin{array}{ccc\|c} 1 & 3 & -4 & 8 \\ 0 & 1 & -11 & 5 \\ 0 & 0 & 0 & 4 \end{array}\right]$			
Ex8: $\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$			

Solve the following System of Equations

Ex9:

$$\begin{aligned} 2x + 4y - 6z &= 10 \\ x - y + z &= 4 \\ x + 2y - 3z &= 7 \end{aligned}$$

$$\begin{aligned} -3x + y + z &= 1 \\ \text{Ex10:}\quad 4x - 3y + 2z &= 2 \\ 5x - 3y + z &= 1 \end{aligned}$$