Conditional Probability:

Sometimes additional information is known or assumed about the outcome of an experiment.

This additional information may have an effect on the probabilities of events.

The new probability of an event *E* assuming that event *F* will occur is denoted by $P(E|F)$ and is called the conditional probability that *E* will occur given that *F* will occur.

Conditional probabilities are calculated using either the Reduced Sample Space Method or the Conditional Probability Formula.

Examples:

1. A fair coin is flipped twice.

$$S=\{HH,HT,TH,TT\}$$

$E=\{HH,HT,TH\}$, $F=\{HT,TH,TT\}$, $G=\{HH,TT\}$, **and** $J=\{HT,TH\}$.

Find $P(E),P(F),P(G),P(J)$.

$$P(E)=\frac{3}{4},P(F)=\frac{3}{4},P(G)=\frac{1}{2},P(J)=\frac{1}{2}$$

Find $P(E \mid F)$ using Reduced Sample Space and Conditional Probability Formula.

In an equally likely sample space, $P(A \mid B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{\text{\# of outcomes in } A \cap B}{\text{\# of outcomes in } B}$**.**

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)} = \frac{2}{3}$$

In general, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$**.**

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3}$$

Find the following conditional probabilities using the Reduced Sample Space Method.

$P(F|E)$

$$\frac{n(E \cap F)}{n(E)} = \frac{2}{3}$$

$P(E|G)$

$$\frac{n(E \cap G)}{n(G)} = \frac{1}{2}$$

$P(G|E)$

$$\frac{n(E \cap G)}{n(E)} = \frac{1}{3}$$

$P(G|J)$

$$\frac{n(G \cap J)}{n(J)} = \frac{0}{2} = 0$$

$P(F|J)$

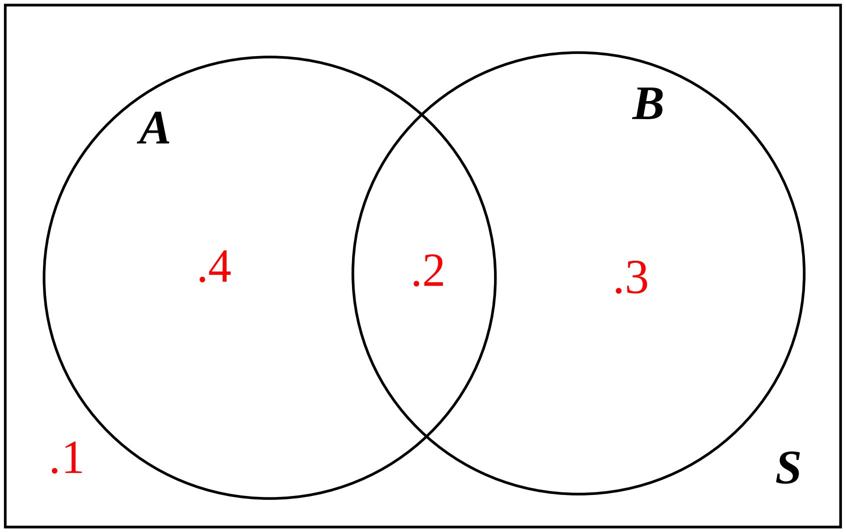
$$\frac{n(F \cap J)}{n(J)} = \frac{2}{2} = 1$$

$P(J|F)$

$$\frac{n(F \cap J)}{n(F)} = \frac{2}{3}$$

2. $P(A) = .6$, $P(B) = .5$, **and** $P(A \cap B) = .2$.

a) Complete the probability diagram.



b) Find $P(A|B)$

$$\frac{P(A \cap B)}{P(B)} = \frac{.2}{.5} = \boxed{\frac{2}{5}}$$

$P(B|A)$

$$\frac{P(A \cap B)}{P(A)} = \frac{.2}{.6} = \boxed{\frac{1}{3}}$$

$P(A|B')$

$$\frac{P(A \cap B')}{P(B')} = \frac{.4}{.5} = \boxed{\frac{4}{5}}$$

$P(B'|A)$

$$\frac{P(A \cap B')}{P(A)} = \frac{.4}{.6} = \boxed{\frac{2}{3}}$$

In this example, we must use the conditional probability formula to calculate these conditional probabilities.

3. Two cards are randomly drawn from a standard 52-card deck in succession without replacement.

a) $P(\text{second card is a heart} \mid \text{the first card is a heart})$

$$\frac{12}{51} = \frac{4}{17}$$

In this example, the conditional probabilities are easily calculated using the Reduced Sample Space method.

b) $P(\text{second card is a heart} \mid \text{the first card is a club})$

$$\frac{13}{51}$$

4. A survey of North Harris students had the following results.

In this example, the conditional probabilities are easily calculated using the Reduced Sample Space method.

	Pepperoni	Sausage	Mushroom	Total
Freshman	25	15	5	45
Sophomore	30	20	5	55
Total	55	35	10	100

A student from the survey is *selected at random*.

a) $P(\text{freshman} \mid \text{pepperoni})$

$$\frac{25}{55} = \frac{5}{11}$$

b) $P(\text{pepperoni} \mid \text{freshman})$

$$\frac{25}{45} = \frac{5}{9}$$

c) $P(\text{sophomore} \mid \text{sausage})$

$$\frac{20}{35} = \frac{4}{7}$$

d) $P(\text{sausage} \mid \text{sophomore})$

$$\frac{20}{55} = \frac{4}{11}$$

Product Rule:

$$P(B|A)=\frac{P(A\cap B)}{P(A)}\text{, so}$$

$$P(A\cap B)=P(A)\cdot P(B|A)$$

Examples:

Two cards are drawn at random from a standard 52-card deck in succession without replacement. Find

$$P(\text{first is a heart and the second is a heart})$$

$$=P(\text{first is a heart})\cdot P(\text{second is a heart}|\text{first is a heart})$$

$$=\frac{13}{52}\cdot\frac{12}{51}=\frac{1}{17}$$

$$P(\text{first is a club and the second is a heart})$$

$$=P(\text{first is a club})\cdot P(\text{second is a heart}|\text{first is a club})$$

$$=\frac{13}{52}\cdot\frac{13}{51}=\frac{13}{204}$$

$$P(\text{first is a club and the second is a heart and the third is a heart})$$

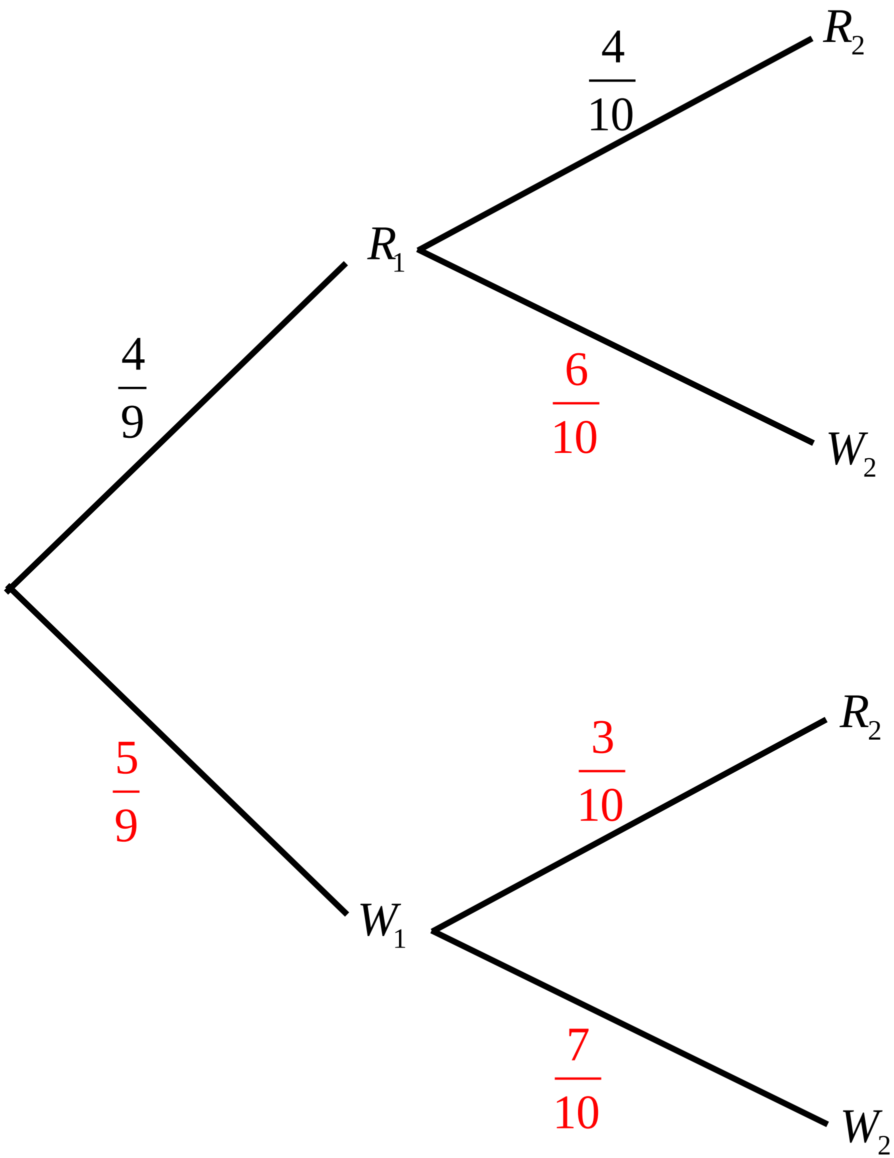
$$\frac{13}{52}\cdot\frac{13}{51}\cdot\frac{12}{50}=\frac{13}{850}$$

The Product Rule also works for three or more events.

Probability Trees:

They keep track of the probabilities and conditional probabilities in a multistage experiment.

Example: There are two bowls: bowl 1 has 4 red marbles and 5 white marbles, and bowl 2 has 3 red marbles and 6 white marbles. A marble is randomly selected from bowl 1 and transferred into bowl 2. Then a marble is randomly selected from bowl 2.


R_2
$\frac{4}{10}$
R_1
$\frac{4}{9}$
$\frac{6}{10}$
W_2
R_2
$\frac{5}{9}$
$\frac{3}{10}$
W_1
$\frac{7}{10}$
W_2

Find

$P(R_1 \cap R_2)$

$$\frac{4}{9}\cdot\frac{4}{10}=\boxed{\frac{16}{90}}$$

$P(R_2)$

$$\frac{4}{9}\cdot\frac{4}{10}+\frac{5}{9}\cdot\frac{3}{10}=\boxed{\frac{31}{90}}$$

$P(W_2)$

$$1-P(R_2)=1-\frac{31}{90}=\boxed{\frac{59}{90}}$$

or

$$\frac{4}{9}\cdot\frac{6}{10}+\frac{5}{9}\cdot\frac{7}{10}=\boxed{\frac{59}{90}}$$

$P(R_1 \mid R_2)$

$$\frac{P(R_1 \cap R_2)}{P(R_2)}=\frac{\frac{16}{90}}{\frac{31}{90}}=\boxed{\frac{16}{31}}$$

$P(W_1 \mid R_2)$

$$1-P(R_1 \mid R_2)=1-\frac{16}{31}=\boxed{\frac{15}{31}}$$

or

$$\frac{P(W_1 \cap R_2)}{P(R_2)}=\frac{\frac{15}{90}}{\frac{31}{90}}=\boxed{\frac{15}{31}}$$

$P(R_1 \mid W_2)$

$$\frac{P(R_1 \cap W_2)}{P(W_2)}$$

$$=\frac{\frac{4}{9}\cdot\frac{6}{10}}{\frac{4}{9}\cdot\frac{6}{10}+\frac{5}{9}\cdot\frac{7}{10}}=\frac{\frac{24}{90}}{\frac{59}{90}}=\boxed{\frac{24}{59}}$$

$$P\left(W_1 \mid W_2\right)$$

$$1 - P\left(R_1 \mid W_2\right) = 1 - \frac{24}{59} = \boxed{\frac{35}{59}}$$

or

$$\frac{P\left(W_1 \cap W_2\right)}{P\left(W_2\right)} = \frac{\frac{35}{90}}{\frac{59}{90}} = \boxed{\frac{35}{59}}$$