Inverse Relations:

The inverse of a relation is the relation you get when you interchange the numbers in the ordered pairs.

$R=\{(1,2),(2,2),(3,4)\}$ **Domain?** $\{1,2,3\}$ **Range?** $\{2,4\}$

$S=\{(1,1),(2,3),(3,4)\}$ **Domain?** $\{1,2,3\}$ **Range?** $\{1,3,4\}$

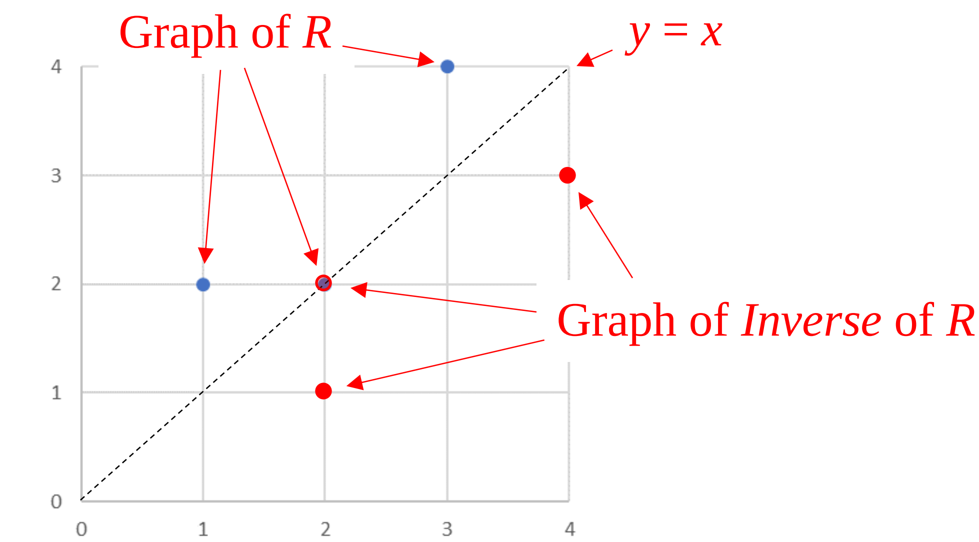
$\textit{Inverse}\text{ of }R=\{(2,1),(2,2),(4,3)\}$ **Domain?** $\{2,4\}$ **Range?** $\{1,2,3\}$

$\textit{Inverse}\text{ of }S=\{(1,1),(3,2),(4,3)\}$ **Domain?** $\{1,3,4\}$ **Range?** $\{1,2,3\}$

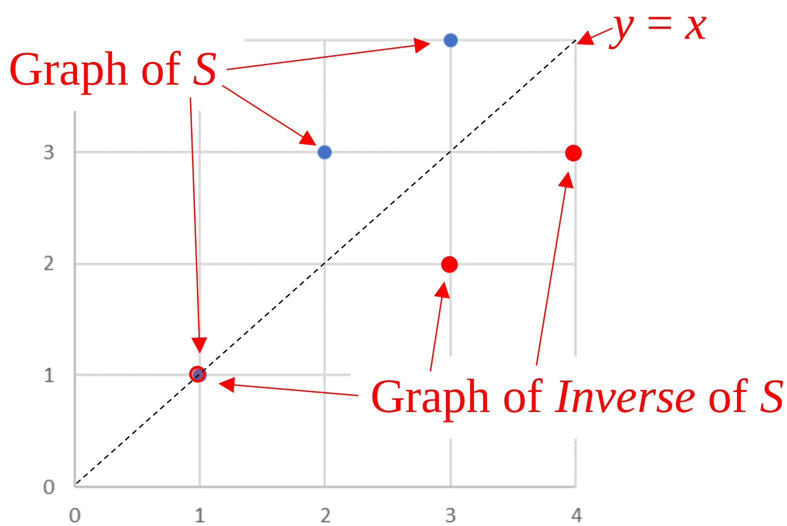
Notice the reversal of Domain and Range between relation and inverse relation.

What's the connection between the graphs of relations and their inverses?

Check them out.



The graph of the inverse relation is the reflection, with respect to the line $y = x$, of the graph of the relation.



When the relation *f*, is a function, and its inverse is also a function, then the function *f* is said to be invertible, and there is a special notation for its inverse function, f^{-1}.

Is R invertible? No, its inverse relation is not a function.

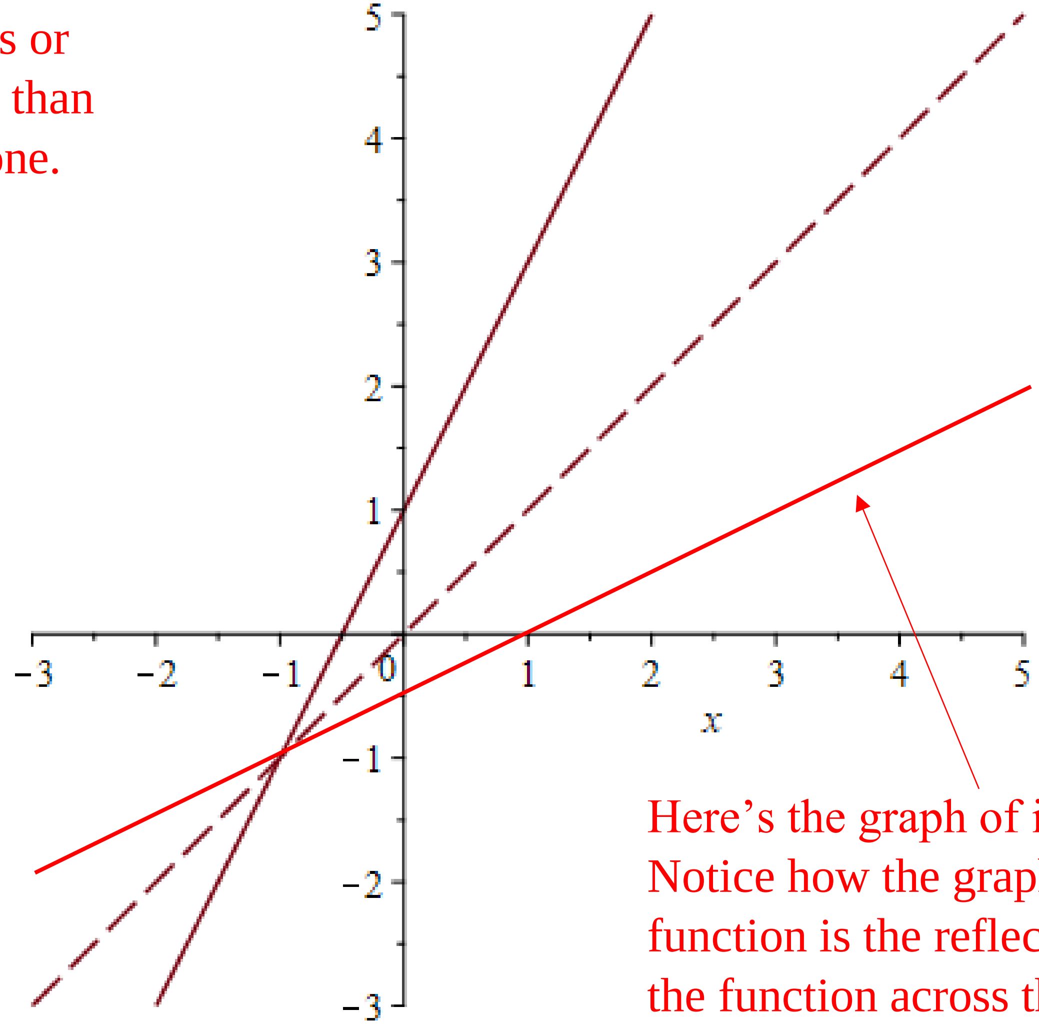
Is S invertible? Yes, both S and its inverse are functions.

In order for the inverse of a function to also be a function, all of the range values of the original function must be unique. Functions that have this property are called one-to-one functions. A function *f* is one-to-one if whenever $f(x) = f(y)$, then $x = y$.

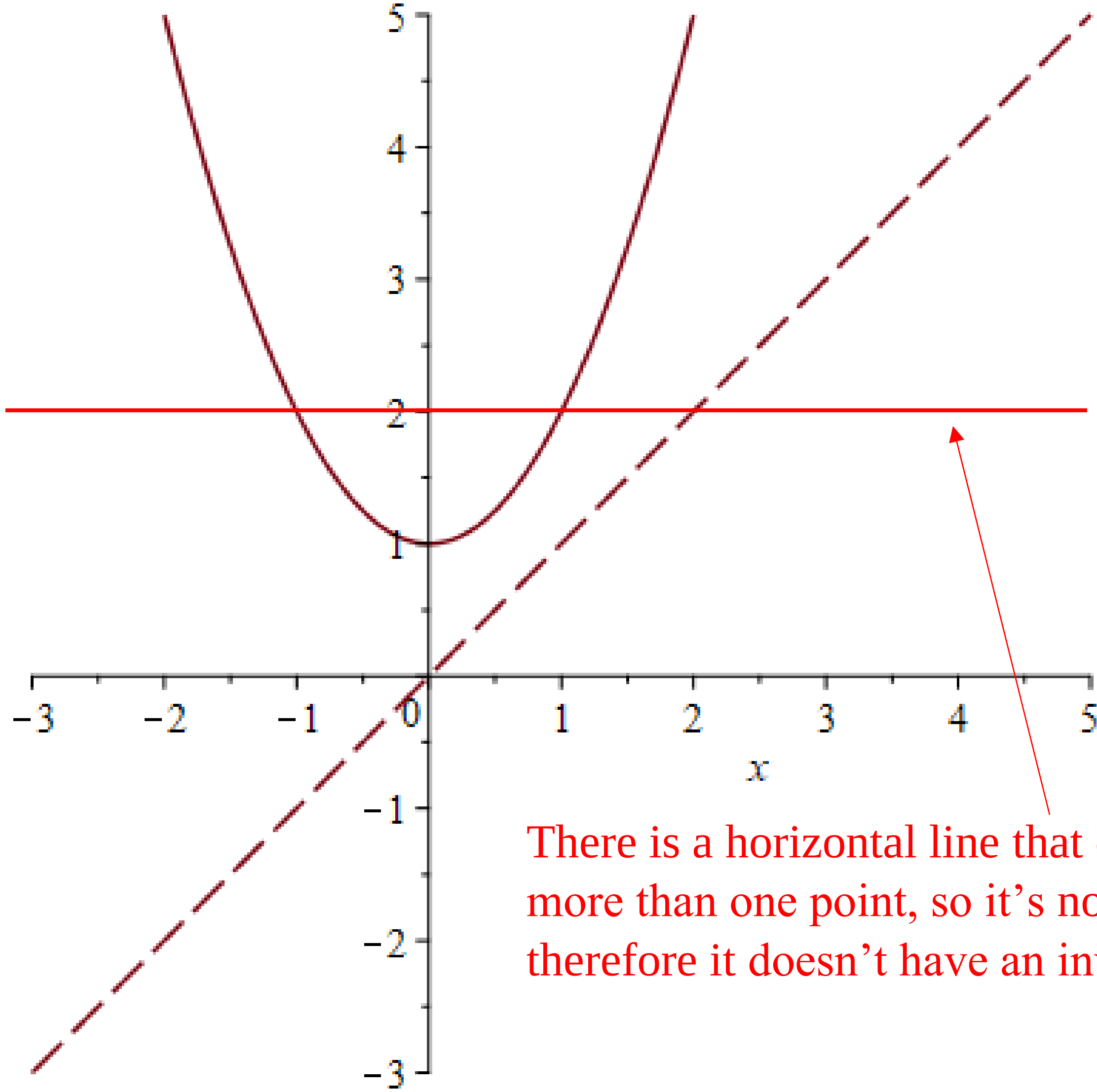
One-to-one functions are invertible, and there is a graphical test for one-to-oneness called the Horizontal Line Test-If no horizontal line touches or crosses the graph in more than one point, then the function is one-to-one; otherwise, it's not.

Determine if the following functions are one-to-one, and therefore have an inverse function. Graph the inverse function, as well.

No horizontal line touches or crosses the graph in more than one point, so it's one-to-one.



Here's the graph of its inverse function. Notice how the graph of the inverse function is the reflection of the graph of the function across the line $y = x$.



There is a horizontal line that crosses the graph in more than one point, so it’s not one-to-one, and therefore it doesn’t have an inverse function.

Finding formulas for inverse functions:

Sometimes you can eyeball the function formula and find a formula for the inverse function:

1. $f(x) = 2x$

First let's show that this function is one-to-one. If we graph it, we could use the Horizontal Line Test. If we suppose that $2x = 2y$, then dividing by 2 leads to $x = y$, and this means it's one-to-one. The opposite of multiplying by 2 is dividing by 2, or multiplying by $\frac{1}{2}$, so $\boxed{f^{-1}(x) = \frac{1}{2}x}$.

2. $f(x) = x - 1$

First let's show that this function is one-to-one. If we graph it, we could use the Horizontal Line Test. If we suppose that $x - 1 = y - 1$, then adding 1 leads to $x = y$, and this means it's one-to-one. The opposite of subtracting 1 is adding 1, so $\boxed{f^{-1}(x) = x + 1}$.

3. $f(x) = 3x + 1$

First let's show that this function is one-to-one. If we graph it, we could use the Horizontal Line Test. If we suppose that $3x + 1 = 3y + 1$, then subtracting 1 and dividing by 3 leads to $x = y$, and this means it's one-to-one. The opposite of multiplying by 3 and adding 1 is subtracting 1 and dividing by 3, so $\boxed{f^{-1}(x) = \frac{x-1}{3}}$.

There is a definite procedure for finding a formula for an inverse function.

1. Replace $f(x)$ with *y*.

2. Interchange *x* and *y*.

3. Solve for *y*.

4. Replace y with $f^{-1}(x)$.

Examples:

1. $f(x) = 3x + 1$

$$y = 3x + 1$$

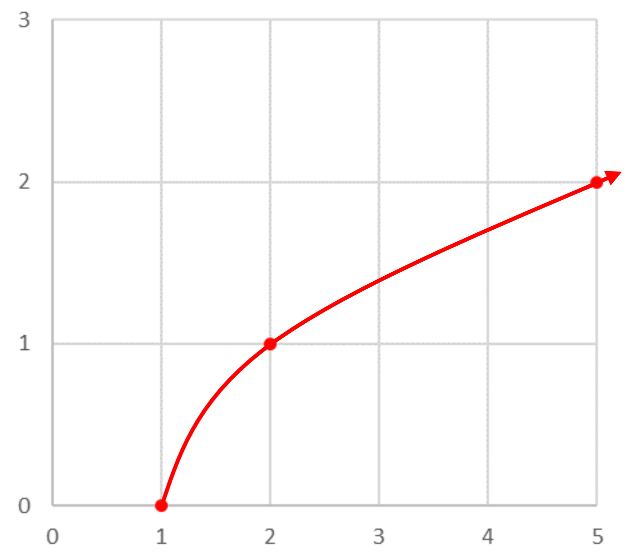
$$x = 3y + 1 \Rightarrow x - 1 = 3y \Rightarrow \frac{x-1}{3} = y$$

$$\Rightarrow \boxed{f^{-1}(x) = \frac{x-1}{3}}$$

2. $f(x)=\sqrt{x-1}$

From the graph of *f*, we see that it's one-to-one, its domain is $[1,\infty)$ and its range is $[0,\infty)$

.



$$y=\sqrt{x-1}$$

$$x=\sqrt{y-1}\Rightarrow x^2=y-1\Rightarrow x^2+1=y$$

$$\Rightarrow f^{-1}(x)=x^2+1$$

This is not the formula of a one-to-one function, so we'll have to restrict its domain to the range of the original function- $\boxed{f^{-1}(x)=x^2+1;x\geq 0}$.

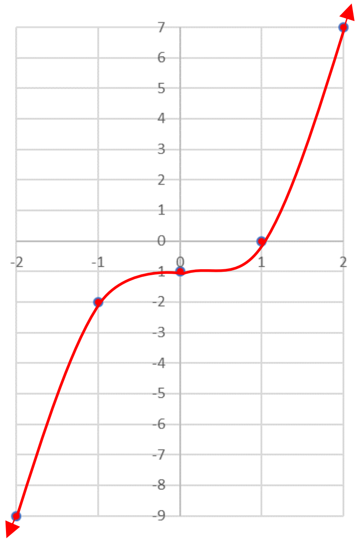
3. $f(x) = x^3 - 1$

From the Horizontal Line Test applied to the graph, *f* is one-to-one, and therefore has an inverse function.

$$y = x^3 - 1$$

$$x = y^3 - 1 \Rightarrow x + 1 = y^3 \Rightarrow \sqrt[3]{x+1} = y$$

$$\boxed{f^{-1}(x) = \sqrt[3]{x+1}}$$



4. $f(x)=\dfrac{x+4}{x-3}$

Suppose that $\dfrac{x+4}{x-3}=\dfrac{y+4}{y-3}\Rightarrow xy-3x+4y-12=xy-3y+4x-12$

$\Rightarrow 3y-3x+4y-4x=0\Rightarrow 7(y-x)=0$

$\Rightarrow x=y\Rightarrow f$ is one-to-one.

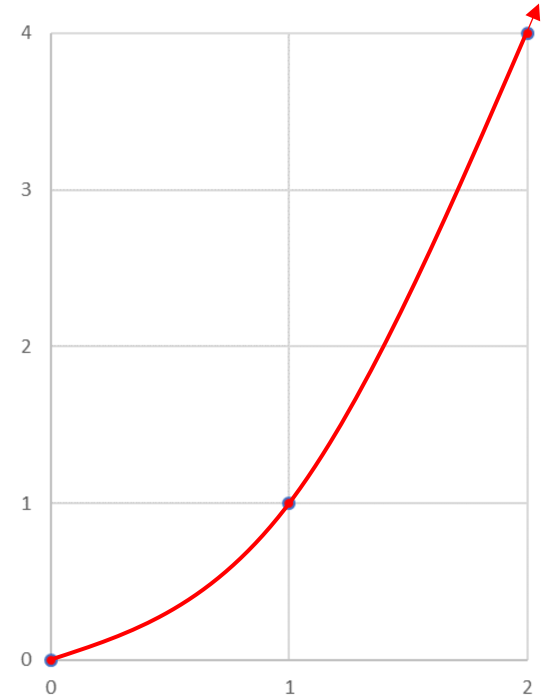
$$y=\frac{x+4}{x-3}$$

$$x=\frac{y+4}{y-3}\Rightarrow xy-3x=y+4\Rightarrow xy-y=3x+4$$

$$\Rightarrow y(x-1)=3x+4\Rightarrow y=\frac{3x+4}{x-1}$$

$$\boxed{f^{-1}(x)=\frac{3x+4}{x-1}}$$

5. $f(x) = x^2; x \geq 0$



From the graph of *f*, we can see that it's one-to-one, and its domain and range are both $[0, \infty)$.

$$y = x^2$$

$$x = y^2 \Rightarrow y = \pm\sqrt{x}$$

We know that the range of the inverse function must be $[0, \infty)$.

$$\boxed{f^{-1}(x) = \sqrt{x}}$$

Composition Property of Inverse Functions:

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = x \text{ for all } x \text{ in the domain of } f^{-1}(x)$$

And

$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = x \text{ for all } x \text{ in the domain of } f(x).$$

Example:

For $f = \{(1,2),(2,3)\}$ **and** $f^{-1} = \{(2,1),(3,2)\}$

$f\left(f^{-1}(2)\right) = f(1) = \boxed{2}$ **and** $f\left(f^{-1}(3)\right) = f(2) = \boxed{3}$

$f^{-1}\left(f(1)\right) = f^{-1}(2) = \boxed{1}$ **and** $f^{-1}\left(f(2)\right) = f^{-1}(3) = \boxed{2}$

Are the functions $f(x) = 2x - 1$ **and** $g(x) = \frac{1}{2}x + 1$ **inverses?**

$$(f \circ g)(x) = 2\left(\tfrac{1}{2}x + 1\right) - 1 = x + 2 - 1 = x + 1 \neq x$$

No.

Are the functions $f(x) = \sqrt{x}$ **and** $g(x) = x^2$ **inverses?**

$$(f \circ g)(x) = \sqrt{x^2} = |x| \neq x$$

No.