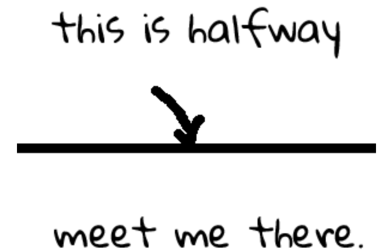
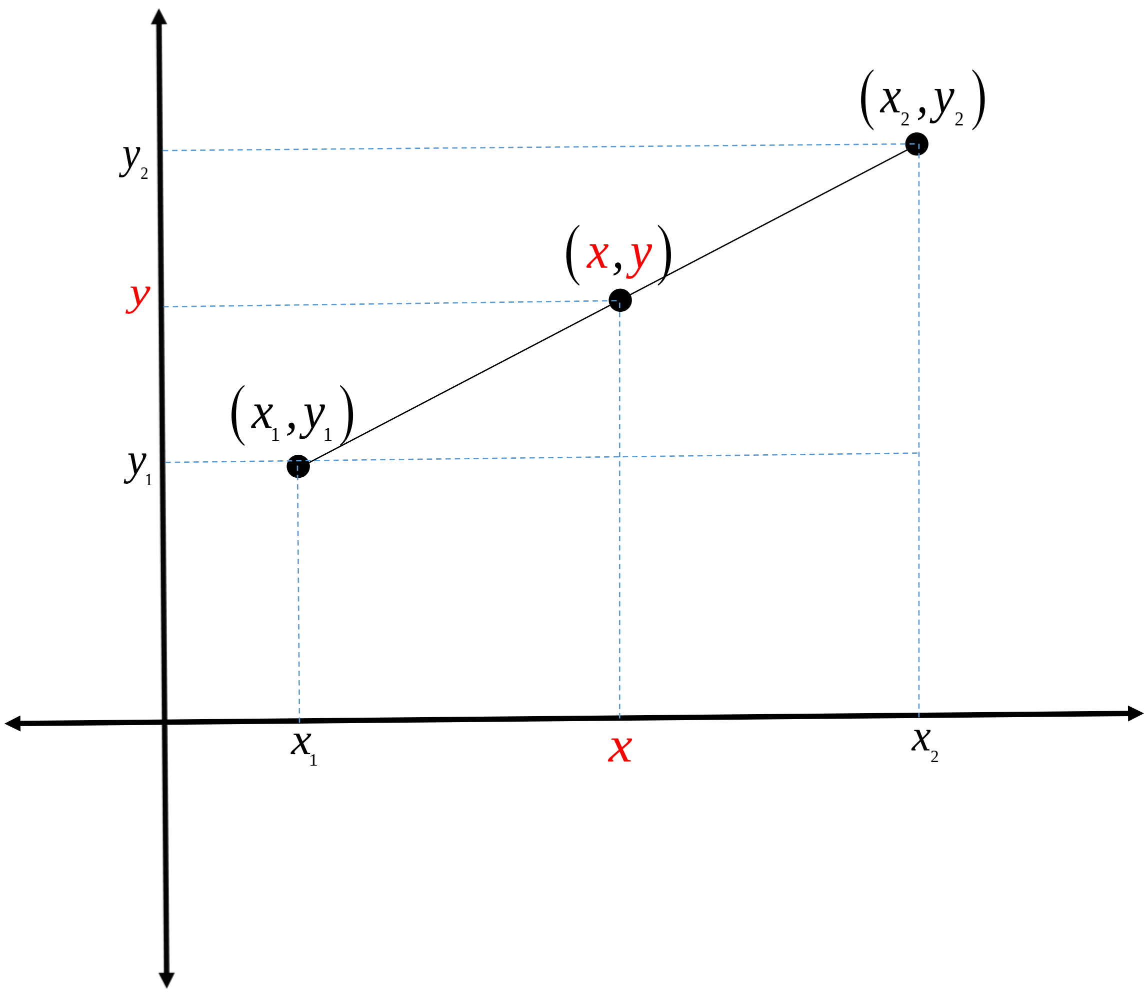
Midpoint Formula:

Given two points (x_1, y_1) and (x_2, y_2), the point on the line segment connecting them that is halfway between them is called their midpoint.





From similar triangles,

$$\frac{x_2 - x_1}{x - x_1} = 2 \Rightarrow x_2 - x_1 = 2x - 2x_1$$

$$\Rightarrow x = \frac{x_1 + x_2}{2}$$

From similar triangles,

$$\frac{y_2 - y_1}{y - y_1} = 2 \Rightarrow y_2 - y_1 = 2y - 2y_1$$

$$\Rightarrow y = \frac{y_1 + y_2}{2}$$

So midpoint $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$.

Examples: Find the midpoints of the following pairs of points.

1. $(1,2)$ and $(3,4)$

$$\left(\frac{1+3}{2}, \frac{2+4}{2} \right) = \boxed{(2,3)}$$

2. $(1,-2)$ and $(-3,4)$

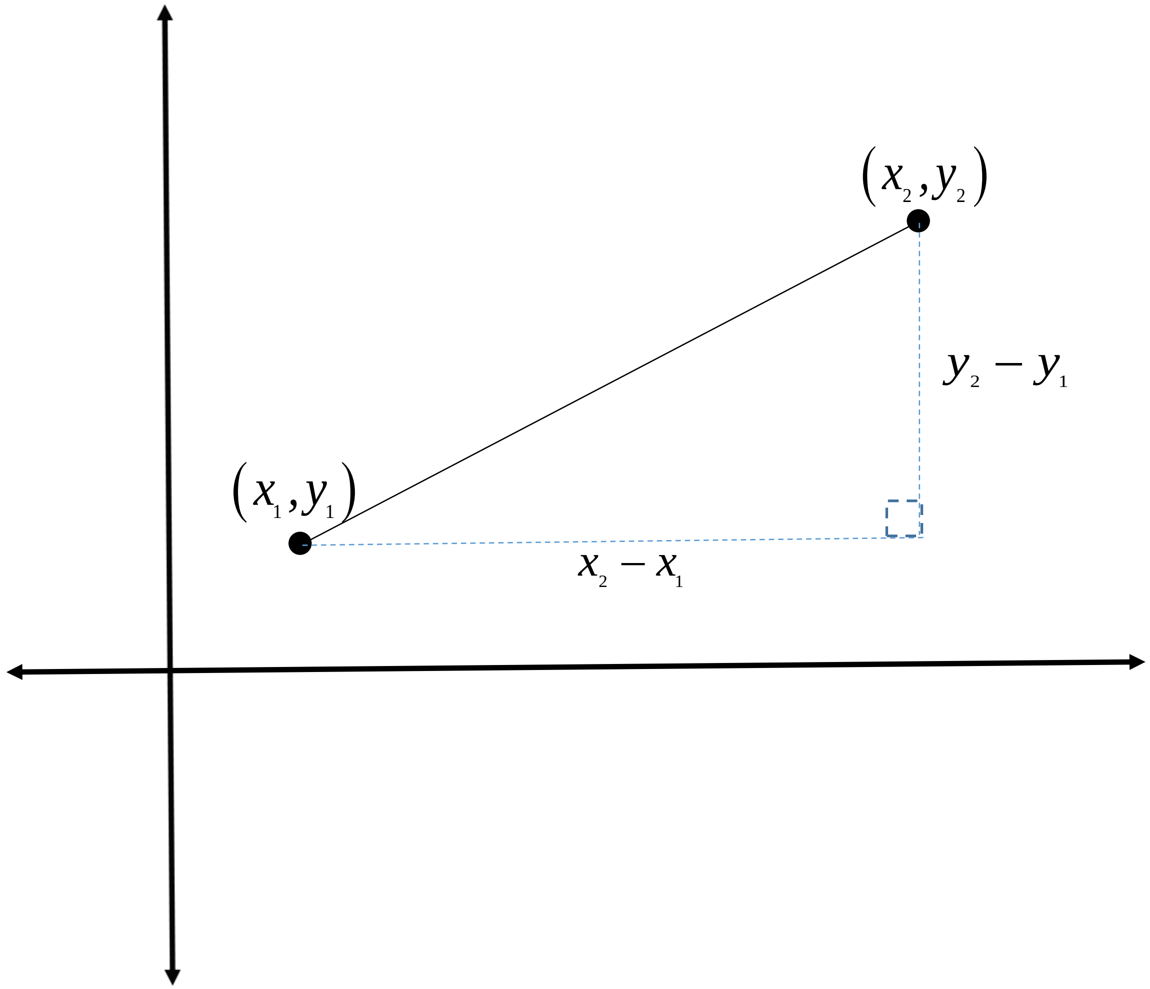
$$\left(\frac{1+(-3)}{2}, \frac{(-2)+4}{2} \right) = \boxed{(-1,1)}$$

3. $(1,-2)$ and $(4,-6)$

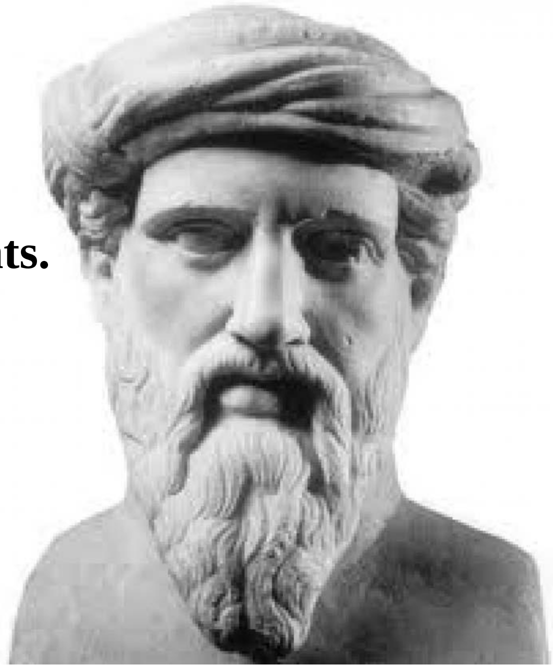
$$\left(\frac{1+4}{2}, \frac{(-2)+(-6)}{2} \right) = \boxed{\left(\tfrac{5}{2},-4\right)}$$

Distance Formula:

Given two points (x_1, y_1) and (x_2, y_2), the length of the line segment connecting them is the distance between the two points.



From the Pythagorean Theorem, we get that $(\text{Distance})^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$**, so we get that** $\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$**.**



Examples: Find the distance between the following pairs of points.

1. $(1,2)$ and $(4,6)$

$\sqrt{(4-1)^2 + (6-2)^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = \boxed{5}$

2. $(1,-2)$ and $(-4,10)$

$\sqrt{(-4-1)^2 + (10--2)^2} = \sqrt{(-5)^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = \boxed{13}$

3. $(1,-2)$ and $(4,-5)$

$\sqrt{(4-1)^2 + (-5--2)^2} = \sqrt{3^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18} = \boxed{3\sqrt{2}}$

Equations of Circles:

Let's find an equation for a circle with center of (h,k) and radius of *r*. It would be the set of points (x,y) whose distance to (h,k) is equal to *r*.

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

$$(x-h)^2+(y-k)^2=r^2$$

This is called the standard form of the equation of a circle.

Examples: Find equations for the circles described.

1. center of $(1,2)$ and radius of 3

$$(x-1)^2+(y-2)^2=3^2 \Rightarrow \boxed{(x-1)^2+(y-2)^2=9}$$

2. center of $(-1,2)$ and radius of $\sqrt{3}$

$$(x--1)^2+(y-2)^2=\left(\sqrt{3}\right)^2 \Rightarrow \boxed{(x+1)^2+(y-2)^2=3}$$

3. center of $(0,-2)$ and radius of 5

$$(x-0)^2+(y--2)^2=5^2 \Rightarrow \boxed{x^2+(y+2)^2=25}$$

4. $(1,2)$ and $(3,4)$ **are the endpoints of a diameter**

The midpoint of the segment connecting these two points must be the center of the circle.

$\left(\frac{1+3}{2},\frac{2+4}{2}\right)=(2,3)\Rightarrow(x-2)^2+(y-3)^2=r^2$, and if we plug either endpoint into this equation, we'll find the value of r^2.

$$(1-2)^2+(2-3)^2=r^2 \Rightarrow r^2=2\Rightarrow \boxed{(x-2)^2+(y-3)^2=2}$$

There is another form of the equation of a circle called the general form.

$$x^2 + y^2 + cx + dy + e = 0$$

The graph of the solutions of an equation of this form can be a circle, a point, or nothing at all.

To determine the nature of the graph, you complete the square in both *x* and *y* to convert the equation to standard form.

$$(x-h)^2 + (y-k)^2 = c$$

If $c > 0$, then the graph is a circle.

If $c = 0$, then the graph is a point.

If $c < 0$, then there is no graph.

Examples: Determine if the graph of the given equation is a circle, a point, or nothing at all. If it's a circle, give its center and radius. If it's a point give the coordinates.

1. $x^2 + y^2 - 8x + 2y - 19 = 0$

$(x^2 - 8x) + (y^2 + 2y) = 19 \Rightarrow (x^2 - 8x + 16) + (y^2 + 2y + 1) = 19 + 16 + 1$

$\Rightarrow (x-4)^2 + (y+1)^2 = 36 \Rightarrow$ The graph is a circle centered at $(4,-1)$ with radius 6.

2. $x^2 + y^2 + 2x - 6y + 10 = 0$

$(x^2 + 2x) + (y^2 - 6y) = -10 \Rightarrow (x^2 + 2x + 1) + (y^2 - 6y + 9) = -10 + 1 + 9$

$\Rightarrow (x+1)^2 + (y-3)^2 = 0 \Rightarrow$ The graph is the single point $(-1,3)$.

3. $x^2 + y^2 - 4x - 8y + 21 = 0$

$(x^2 - 4x) + (y^2 - 8y) = -21 \Rightarrow (x^2 - 4x + 4) + (y^2 - 8y + 16) = -21 + 4 + 16$

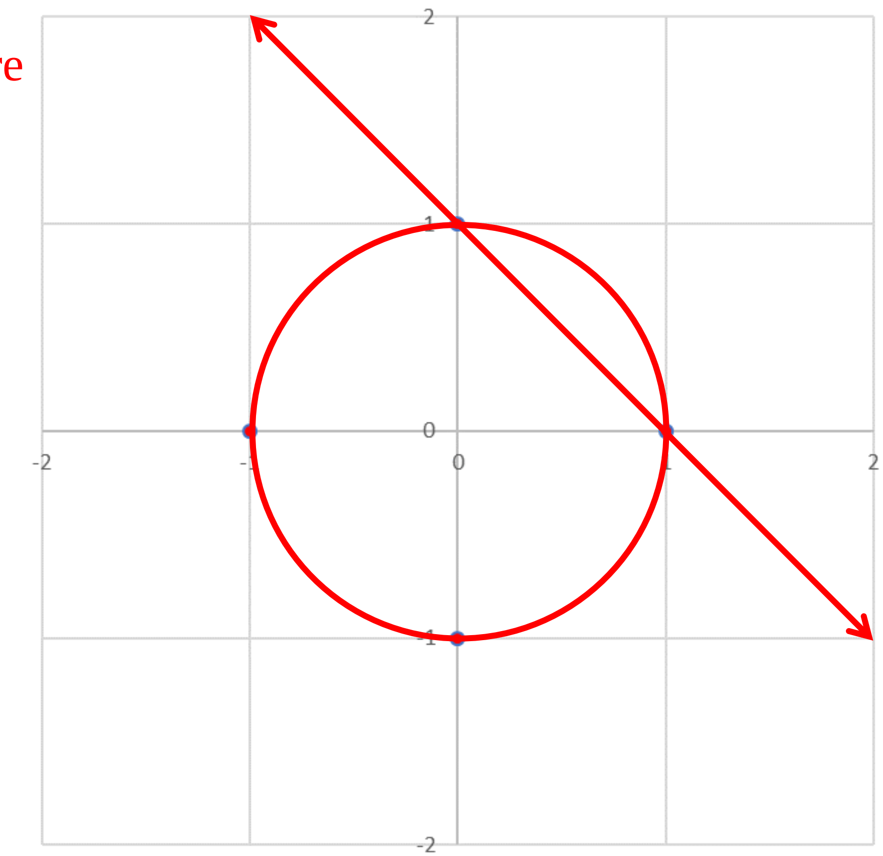
$\Rightarrow (x-2)^2 + (y-4)^2 = -1 \Rightarrow$ There is no graph.

Points of Intersection:

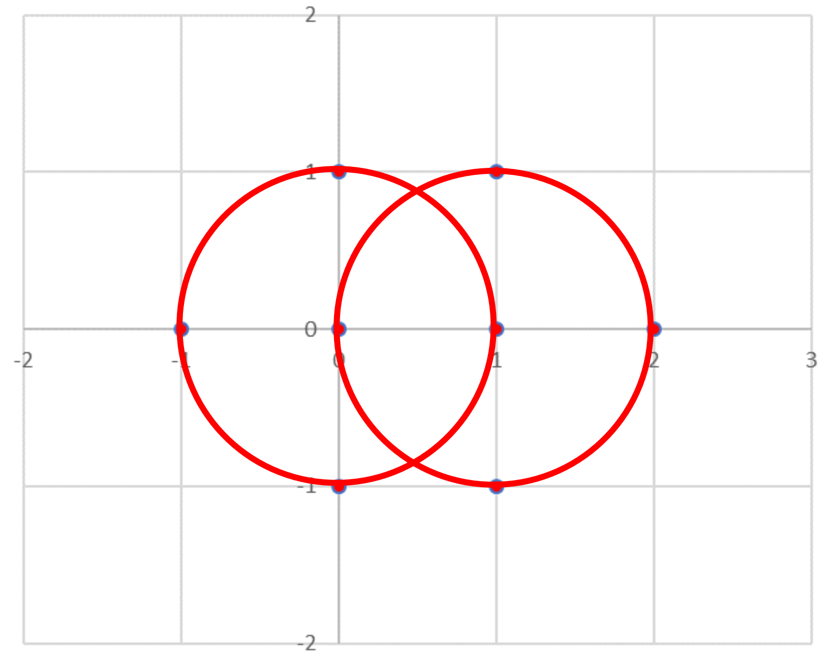
Find the points of intersection of the graphs of the solutions of the given pair of equations.

1. $x^2 + y^2 = 1$ and $x + y = 1$

From the graph, you can see that there are two points of intersection, and they are $(1,0)$ and $(0,1)$.



2. $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$



From the graph, you can see that there Are two points of intersection. To find Their coordinates, we'll have to solve the system of equations $\begin{matrix} x^2 + y^2 = 1 \\ (x-1)^2 + y^2 = 1 \end{matrix}$.

Subtracting the first equation from the second leads to $(x-1)^2 - x^2 = 0 \Rightarrow -2x + 1 = 0 \Rightarrow x = \frac{1}{2}$.

Plugging this into the first equation leads to $\frac{1}{4} + y^2 = 1 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$. So the points of intersection are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.