Systems of Linear Equations as Augmented Matrices:

The original linear system

$$\begin{aligned} 2x + 3y &= -1 \\ 3x - 4y &= 7 \end{aligned}$$

The linear system represented as an augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & -1 \\ 3 & -4 & 7 \end{array}\right]$$

The horizontal arrangements of numbers are called rows, and the vertical arrangements are called columns. The first column represents the coefficients of the *x* variable, and the second column represents the coefficients of the *y* variable. The vertical bar abbreviates the equal signs.

Example:

Convert the system of equations

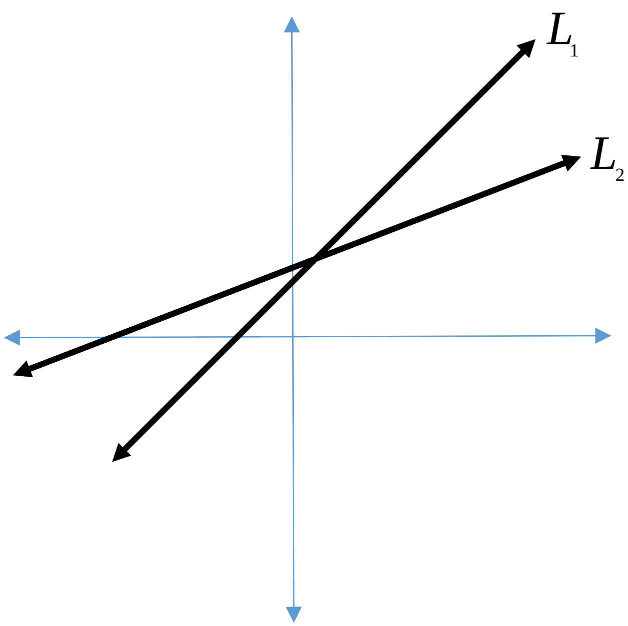
$$\begin{aligned} x - 2y + z &= 2 \\ -3x + y - 3z &= 7 \end{aligned}$$

into augmented matrix form.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -3 & 1 & -3 & 7 \end{array}\right]$$

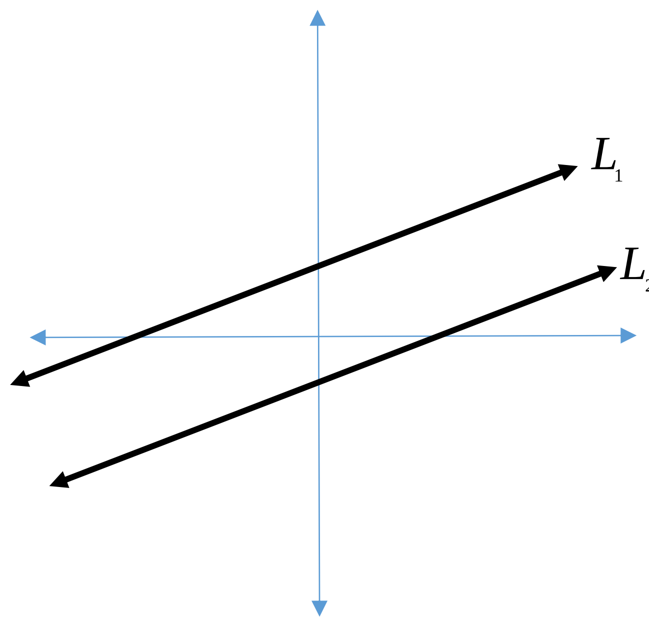
In the case of two equations and two variables, there are three possibilities for solution(s) of the system.

Possibilities:



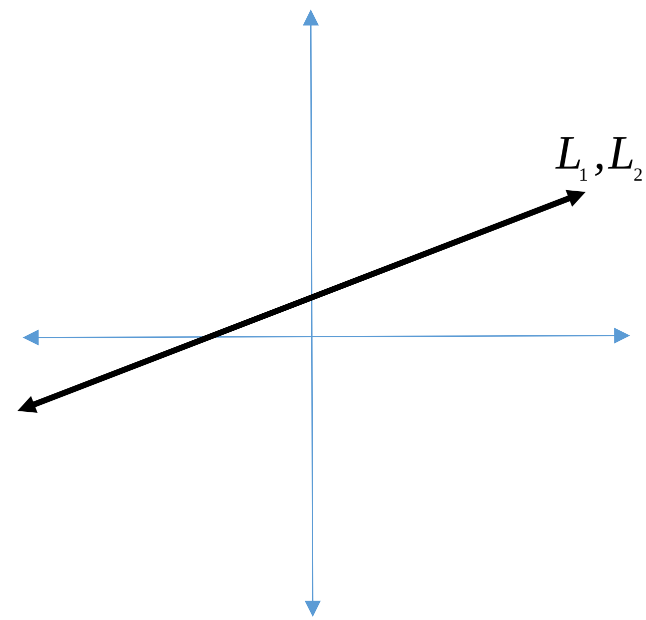
1 solution

Corresponding to the single point of intersection of the solutions lines



No solution

Due to the fact that the solution lines have no points in common



Infinitely many solutions

Due to the fact that every point on the common solution line corresponds to a solution of the system

These three possibilities are true for any number of equations and any number of variables.

The method of solving systems called elimination by addition is carried out on the augmented matrix. The rows of the matrix are treated as equations. The following operations can be performed on the rows of the matrix without changing the solution(s).

<u>Row Operations:</u>

1. Two rows can be interchanged.

2. A row can be multiplied by a non-zero number.

3. A multiple of one row can be added to another row.

The goal is to reach a matrix with the following form.

$$\left[\begin{array}{cccccc|c} 1 & & & & & & * \\ 0 & 1 & & & & & \% \\ 0 & 0 & 1 & & & & \$ \\ \vdots & \vdots & 0 & 1 & & & \# \\ \vdots & \vdots & \vdots & \vdots & & & \wp \\ 0 & 0 & 0 & 0 & & & \Diamond \end{array}\right]$$

As many 1's as possible on the diagonal with zeros below the 1's.

The process of using row operations to reach the goal is called Gaussian Elimination.

Examples:

1. $\begin{aligned} 2x+3y&=-1 \\ 3x-4y&=7 \end{aligned}$

$$\left[\begin{array}{cc|c} 2 & 3 & -1 \\ 3 & -4 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -4 & 7 \\ 2 & 3 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 2 & 3 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 17 & -17 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 1 & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ — This abbreviates switching row 1 and row 2.

$-R_2 + R_1 \to R_1$ — This abbreviates adding -1 times row 2 to row 1.

$-2R_1 + R_2 \to R_2$

$\frac{1}{17}R_2 \to R_2$ — This abbreviates multiplying row 2 by $\frac{1}{17}$.

Now that the goal has been reached, convert the last row back into an equation:

$$y=-1.$$

Convert the first row back into an equation:

$$x-7y=8,$$

and substitute the previously determined value of y into this equation.

$$x-7(-1)=8 \Rightarrow x+7=8 \Rightarrow x=1$$

So the system has one solution, and it's $x=1$ and $y=-1$.

The process of starting at the bottom of the final matrix and working toward the top is called back substitution.

2. $$\begin{aligned} x + y &= 1 \\ -2x - 2y &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

$2R_1 + R_2 \to R_2$

Now that the goal has been reached, convert the last row back into an equation:

$$0 = 4.$$

Since this is impossible, the system has no solution. If at any time in the process of reaching the goal, you get a row with zeros to the left of the bar and a non-zero number to the right, you may stop and conclude that the system has no solution.

3. $$\begin{aligned} x + y &= 1 \\ 3x + 3y &= 3 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 3 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$-3R_1 + R_2 \rightarrow R_2$

Now that the goal has been reached, convert the last row back into an equation:

$$0 = 0.$$

There is no contradiction, and we can't uniquely solve for the values of the variables from the first row(equation): $x + y = 1$.

When this happens, the system has infinitely many solutions, and we represent them as follows:

Let *y* be an arbitrary real number, and solve for *x* in the first equation in terms of *y* to get $x + y = 1 \Rightarrow x = 1 - y$.

The solutions of the system are given by $x = 1 - y, y = y$; where y is any real number.

4. $$\begin{aligned} x-2y+z&=2\\ -3x+y+2z&=4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2\\ -3 & 1 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2\\ 0 & -5 & 5 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2\\ 0 & 1 & -1 & -2 \end{array}\right]$$

$3R_1 + R_2 \rightarrow R_2$ $\qquad -\frac{1}{5}R_2 \rightarrow R_2$

Now that the goal has been reached, notice that there are no contradictions, and we can't uniquely solve for the values of the variables. The system has infinitely many solutions, and we'll represent them by letting the variable furthest to the right, *z*, be an arbitrary real number.

Solve for *y* in the last row(equation) in terms of *z* to get $y-z=-2 \Rightarrow y=z-2$.

Substitute the *y* value into the first row(equation) and solve for *x* in terms of *z* to get

$x-2y+z=2 \Rightarrow x-2(z-2)+z=2 \Rightarrow x-z+4=2 \Rightarrow x=z-2$.

The solutions of the system are given by

$x=z-2, y=z-2, z=z$; where z is any real number.

More Examples:

1. $\begin{matrix} x-2y=1 \\ 2x-y=5 \end{matrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -1 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$-2R_1 + R_2 \to R_2 \quad \frac{1}{3}R_2 \to R_2$

From the last row, we know that $y=1$, and substituting into the first row leads to $x-2=1$, which implies that $x=3$. So the only solution of this system is $x=3$ and $y=1$.

2.
$$\begin{aligned} x + 2y &= 4 \\ 2x + 4y &= -8 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & -8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & -16 \end{array}\right]$$

$$-2R_1 + R_2 \rightarrow R_2$$

Since the last row is equivalent to $0 = -16$, we know that this system has no solution.

3.
$$\begin{aligned} 3x - 6y &= -9 \\ -2x + 4y &= 6 \end{aligned}$$

$$\left[\begin{array}{cc|c} 3 & -6 & -9 \\ -2 & 4 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -3 \\ -2 & 4 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{3}R_1 \rightarrow R_1$ $\qquad$ $2R_1 + R_2 \rightarrow R_2$

We've gone as far as possible, there are no contradictions, and we can't uniquely solve for the variables. This means that there are infinitely many solutions. Starting with the variable on the right, let y be an arbitrary real number, and the first row turns into $x - 2y = -3 \Rightarrow x = 2y - 3$. So the solutions are $x = 2y - 3$ and $y = y$; where y is any real number.

$$
\begin{aligned}
2x - y &= 0 \\
\textbf{4. } \; 3x + 2y &= 7 \\
x - y &= -1
\end{aligned}
$$

$$
\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 3 & 2 & 7 \\ 1 & -1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & 2 & 7 \\ 2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 5 & 10 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 5 & 10 \end{array}\right] \rightarrow
$$

$R_1 \leftrightarrow R_3$ $\quad$ $-3R_1 + R_2 \rightarrow R_2$, $-2R_1 + R_3 \rightarrow R_3$ $\quad$ $R_2 \leftrightarrow R_3$ $\quad$ $-5R_2 + R_3 \rightarrow R_3$

$$
\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]
$$

This is as far as we can go, and we can uniquely solve for x and y. Starting at the bottom, we get that $y = 2$, and substituting above it we get $x - 2 = -1 \Rightarrow x = 1$. So the only solution of the system is $x = 1$ and $y = 2$.