Gauss-Jordan Elimination:

There is an extension of Gaussian Elimination called Gauss-Jordan Elimination.

In general, the goal is to use row operations to reach a matrix with the following form:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & \alpha \\ 0 & 1 & 0 & \cdots & 0 & \beta \\ 0 & 0 & 1 & 0 & \vdots & \gamma \\ \vdots & \vdots & 0 & 1 & 0 & \delta \\ \vdots & \vdots & \vdots & \vdots & & \varepsilon \\ 0 & 0 & 0 & 0 & & \phi \end{array}\right]$$

There are as many 1's as possible on the diagonal with zeros both below the 1's and above the 1's.

Examples:

1. $$\begin{aligned} 2x+3y&=-1 \\ 3x-4y&=7 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & 3 & -1 \\ 3 & -4 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -4 & 7 \\ 2 & 3 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 2 & 3 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 17 & -17 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ $\quad -R_2 + R_1 \to R_1$ $\quad -2R_1 + R_2 \to R_2$ $\quad \frac{1}{17}R_2 \to R_2$ $\quad 7R_2 + R_1 \to R_1$

Now that the goal has been reached, you can easily see that the only solution is $x=1$ **and** $y=-1$**.**

2. $$\begin{aligned} x + y &= 1 \\ -2x - 2y &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

$$2R_1 + R_2 \to R_2$$

Now that the goal has been reached, convert the last row back into an equation:

$$0 = 4.$$

Since this is impossible, the system has no solution. If at any time in the process of reaching the goal, you get a row with zeros to the left of the bar and a non-zero number to the right, you may stop and conclude that the system has no solution.

3. $$\begin{aligned} x+y&=1 \\ 3x+3y&=3 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 3 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$-3R_1 + R_2 \rightarrow R_2$

Now that the goal has been reached, convert the last row back into an equation:

$$0=0.$$

There is no contradiction, and we can't uniquely solve for the values of the variables from the first row(equation): $x+y=1$.

When this happens, the system has infinitely many solutions, and we represent them as follows:

Let *y* be an arbitrary real number, and solve for *x* in the first equation in terms of *y* to get $x+y=1 \Rightarrow x=1-y$.

The solutions of the system are given by $x=1-y, y=y$; where y is any real number.

4. $$\begin{aligned} x-2y+z&=2\\ -3x+y+2z&=4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2\\ -3 & 1 & 2 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2\\ 0 & -5 & 5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2\\ 0 & 1 & -1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2\\ 0 & 1 & -1 & -2 \end{array}\right]$$

$3R_1+R_2 \to R_2$ $\qquad -\frac{1}{5}R_2 \to R_2$ $\qquad 2R_2+R_1 \to R_1$

Now that the goal has been reached, notice that there are no contradictions, and we can't uniquely solve for the values of the variables. The system has infinitely many solutions, and we'll represent them by letting the variable furthest to the right, *z*, be an arbitrary real number.

Solve for *y* in the last row(equation) in terms of *z* to get $y-z=-2 \Rightarrow y=z-2$**. Solve for *x* in the first row(equation) in terms of *z* to get** $x-z=-2 \Rightarrow x=z-2$**.**

The solutions of the system are given by

$x=z-2, y=z-2, z=z$; where z is any real number**.**

More Examples:

1. $$\begin{aligned} x - 2y &= 1 \\ 2x - y &= 5 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

$-2R_1 + R_2 \rightarrow R_2$ $\quad \tfrac{1}{3}R_2 \rightarrow R_2$ $\quad 2R_2 + R_1 \rightarrow R_1$

So we can easily see that the only solution of this system is $x = 3$ and $y = 1$.

2. $$\begin{aligned} x+2y&=4 \\ 2x+4y&=-8 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & -8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & -16 \end{array}\right]$$

$-2R_1 + R_2 \rightarrow R_2$

Since the last row is equivalent to $0=-16$, we know that this system has no solution.

3. $\begin{aligned} 3x-6y&=-9 \\ -2x+4y&=6 \end{aligned}$

$$\left[\begin{array}{cc|c} 3 & -6 & -9 \\ -2 & 4 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -3 \\ -2 & 4 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{3}R_1 \rightarrow R_1$ $\quad$ $2R_1 + R_2 \rightarrow R_2$

We've gone as far as possible, there are no contradictions, and we can't uniquely solve for the variables. This means that there are infinitely many solutions. Starting with the variable on the right, let y be an arbitrary real number, and the first row turns into $x-2y=-3 \Rightarrow x=2y-3$. So the solutions are $x=2y-3$ and $y=y$; where y is any real number.

$$\begin{aligned} 2x+4y-10z&=-2 \\ 3x+9y-21z&=0 \\ x+5y-12z&=1 \end{aligned}$$

4.

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & -2 \\ 3 & 9 & -21 & 0 \\ 1 & 5 & -12 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 3 & 9 & -21 & 0 \\ 2 & 4 & -10 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 0 & -6 & 15 & -3 \\ 0 & -6 & 14 & -4 \end{array}\right] \to$$

$R_1 \leftrightarrow R_3$

$-3R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$-\frac{1}{6}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & -6 & 14 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \to$$

$-5R_2 + R_1 \to R_1$

$6R_2 + R_3 \to R_3$

$-R_3 \to R_3$

$\frac{5}{2}R_3 + R_2 \to R_2$

$-\frac{1}{2}R_3 + R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So the only solution of the system is $x = -2, y = 3, z = 1.$

5. $$\begin{aligned} 2x+4y-2z&=2 \\ -3x-6y+3z&=-3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ -3 & -6 & 3 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -3 & -6 & 3 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{2}R_1 \to R_1$ $\quad 3R_1 + R_2 \to R_2$

We can't go any further, there are no contradictions, and we can't uniquely solve for the variables. This means that there are infinitely many solutions, and this time we'll let both z and y be arbitrary real numbers. Solving for x in the first row, we get $x+2y-z=1 \Rightarrow x=z-2y+1$, so the solutions of the system are given by $z=z, y=y, x=z-2y+1$; where z and y are any real numbers.