## ***Applications of Systems of Linear Equations:***

**1. Animals in an experiment are to be fed a strict diet. Each animal should receive 20 grams of protein and 6 grams of fat. The lab tech is able to purchase two food mixes: Mix A has 10% protein and 6% fat; Mix B has 20% protein and 2% fat. How many grams of each mix should be used to get the right diet for one animal? Let's set it up as a linear system of equations:**

**Let *A* = the number of grams of Mix A, and *B* = the number of grams of Mix B.**

$$.10A + .20B = 20 \quad (\text{Protein Equation})$$
$$.06A + .02B = 6 \quad (\text{Fat Equation})$$

**Let's solve the system using Gaussian Elimination.**

$$\left[\begin{array}{cc|c} .10 & .20 & 20 \\ .06 & .02 & 6 \end{array}\right] \xrightarrow[\substack{10R_1 \to R_1 \\ 100R_2 \to R_2}]{} \left[\begin{array}{cc|c} 1 & 2 & 200 \\ 6 & 2 & 600 \end{array}\right] \xrightarrow[-6R_1 + R_2 \to R_2]{} \left[\begin{array}{cc|c} 1 & 2 & 200 \\ 0 & -10 & -600 \end{array}\right] \xrightarrow[-\frac{1}{10}R_2 \to R_2]{} \left[\begin{array}{cc|c} 1 & 2 & 200 \\ 0 & 1 & 60 \end{array}\right]$$

**$B$ = 60 and $A$ + 120 = 200, so $B$ = 60 and $A$ = 80. So the answer to the question is 80 grams of Mix A and 60 grams of Mix B.**

**2. A company wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15, and 20 passengers, respectively, and the leasing costs are $8,000, $14,000, and $16,000, respectively. What's the cheapest way for the company to accomplish its goal?**

**First, we'll figure out all the different combinations of three types of airplanes the company can lease by solving a linear system of equations:**

**Let $x_1 =$ the number of 10 passenger planes, $x_2 =$ the number of 15 passenger planes, and $x_3 =$ the number of 20 passenger planes.**

$$x_1 + x_2 + x_3 = 12 \quad \text{(Plane Equation)}$$

$$10x_1 + 15x_2 + 20x_3 = 220 \quad \text{(Passenger Equation)}$$

**Let's solve this system using Gauss-Jordan Elimination.**

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 10 & 15 & 20 & 220 \end{array}\right] \underset{-10R_1+R_2\to R_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 5 & 10 & 100 \end{array}\right] \underset{\frac{1}{5}R_2\to R_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 20 \end{array}\right] \underset{-R_2+R_1\to R_1}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -8 \\ 0 & 1 & 2 & 20 \end{array}\right]$$

**So the system has infinitely many solutions given by** $x_1 = x_3 - 8, x_2 = 20 - 2x_3, x_3 = x_3$**; where** $x_3$ **is any real number. But,** $x_1, x_2, \text{and } x_3$ **are numbers of airplanes, so they have to be nonnegative whole numbers.**

**So** $\begin{array}{r} x_3 - 8 \geq 0 \\ 20 - 2x_3 \geq 0 \\ x_3 \geq 0 \end{array}$ **and** $x_3$ **must be a whole number. This means that** $\begin{array}{r} 8 \leq x_3 \\ x_3 \leq 10 \\ 0 \leq x_3 \end{array}$**, and if you combine them you get** $8 \leq x_3 \leq 10$ **and** $x_3$ **is a whole number. So instead of infinitely many solutions, we actually get three of them because** $x_3$ **must be 8, 9 or 10. So the three combinations of airplanes that they can lease are** $x_1 = 0, x_2 = 4, x_3 = 8$

**and** $x_1 = 1, x_2 = 2, x_3 = 9$ **and** $x_1 = 2, x_2 = 0, x_3 = 10$**. Now we have to determine which of these three combinations is the cheapest.**

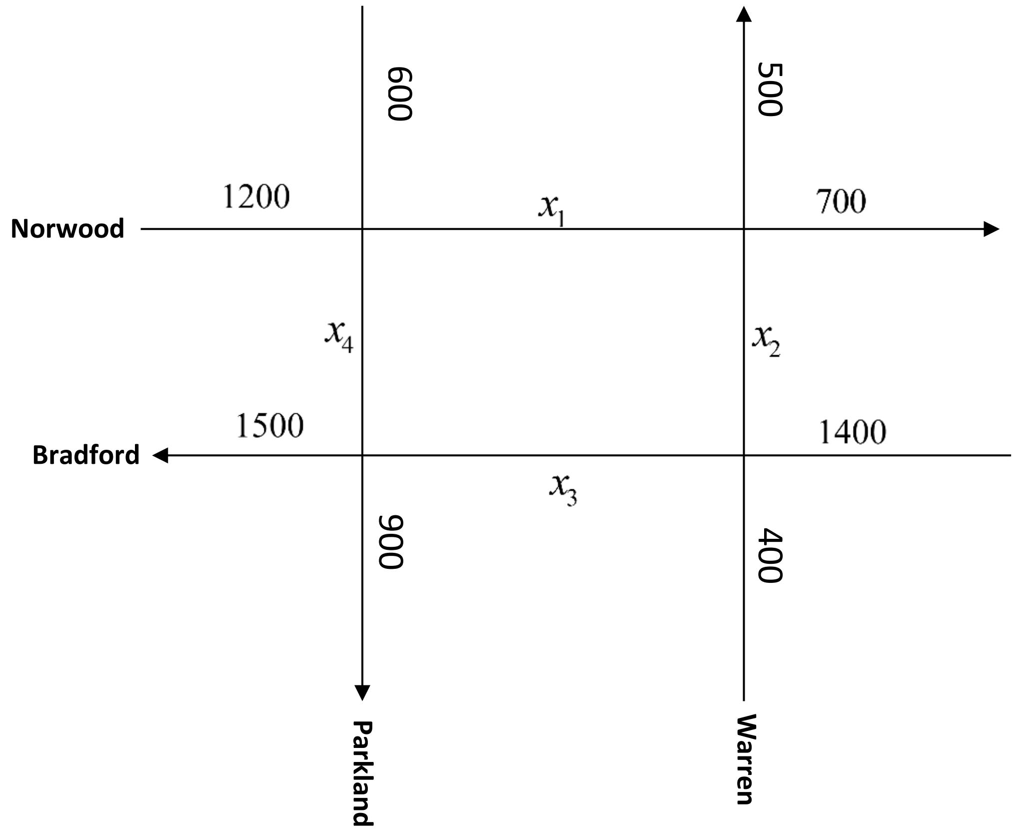
$$x_1 = 0, x_2 = 4, x_3 = 8;\ \text{cost} = \$184,000$$

$$x_1 = 1, x_2 = 2, x_3 = 9;\ \text{cost} = \$180,000$$

$$x_1 = 2, x_2 = 0, x_3 = 10;\ \text{cost} = \$176,000$$

**So the cheapest way for the company to achieve its goal is to lease 2 of the 10 passenger planes and 10 of the 20 passenger planes.**

**3. The diagram shows the traffic flow at the intersections of four one-way streets.**




**The traffic rates are in cars per hour.**

**In order to have smooth traffic flow, the number of cars entering an intersection must equal the number of cars leaving an intersection. This leads to four equations-one for each intersection:**

| Intersection | Equation |
| --- | --- |
| **Norwood and Warren** | $x_1 + x_2 = 1200$ |
| **Bradford and Warren** | $x_2 + x_3 = 1800$ |
| **Bradford and Parkland** | $x_3 + x_4 = 2400$ |
| **Norwood and Parkland** | $x_1 + x_4 = 1800$ |

**Here's the augmented matrix corresponding to the system of linear equations.**

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1200 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2400 \\ 1 & 0 & 0 & 1 & 1800 \end{array}\right]$$

**The result of Gauss-Jordan Elimination is the following matrix:**

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1800 \\ 0 & 1 & 0 & -1 & -600 \\ 0 & 0 & 1 & 1 & 2400 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

**So the mathematical solution is**

$x_4 = x_4, x_3 = 2400 - x_4, x_2 = x_4 - 600, x_1 = 1800 - x_4,$ where $x_4$ is any real #

**Since the traffic flows must be nonnegative, it must be that**

$$\begin{aligned} 0 &\le x_4 \\ x_4 &\le 2400 \\ 600 &\le x_4 \\ x_4 &\le 1800 \end{aligned}$$

**In order for all of these inequalities to be true, it must be that** $600 \le x_4 \le 1800$**. So the true solution of the system is**

$x_4 = x_4, x_3 = 2400 - x_4, x_2 = x_4 - 600, x_1 = 1800 - x_4,$ where $600 \le x_4 \le 1800$.

**Here are the maximum and minimum traffic flows in the network:**

| Street Section | Minimum Flow | Maximum Flow |
|---|---|---|
| **Norwood between Parkland and Warren,** $x_1$ | **0** | **1200** |
| **Warren between Bradford and Norwood,** $x_2$ | **0** | **1200** |
| **Bradford between Warren and Parkland,** $x_3$ | **600** | **1800** |
| **Parkland between Bradford and Norwood,** $x_4$ | **600** | **1800** |

**If traffic on Warren between Bradford and Norwood is restricted to 100 cars per hour due to construction, here's the traffic flow in the rest of the system.**

$$x_2 = x_4 - 600 = 100 \Rightarrow x_4 = 700$$

$$x_3 = 2400 - 700 = 1700, x_1 = 1800 - 700 = 1100$$

**If the following tolls are charged, let's determine the least and greatest amount of money generated from the tolls per hour.**

| Street Section | Toll |
|---|---|
| **Norwood between Parkland and Warren,** $x_1$ | **$.25** |
| **Warren between Bradford and Norwood,** $x_2$ | **$.50** |
| **Bradford between Warren and Parkland,** $x_3$ | **$.20** |
| **Parkland between Bradford and Norwood,** $x_4$ | **$.15** |

**The total toll per hour in cents is**

$$\begin{aligned}
&25x_1 + 50x_2 + 20x_3 + 15x_4 \\
&= 25(1800 - x_4) + 50(x_4 - 600) + 20(2400 - x_4) + 15x_4 \\
&= 20x_4 + 63000;\ 600 \le x_4 \le 1800
\end{aligned}$$

**So the maximum toll amount will occur for** $x_4 = 1800$**, giving a maximum toll amount of 99,000 cents or $990, and the minimum toll amount will occur for** $x_4 = 600$**, giving a minimum toll amount of 75,000 cents or $750.**