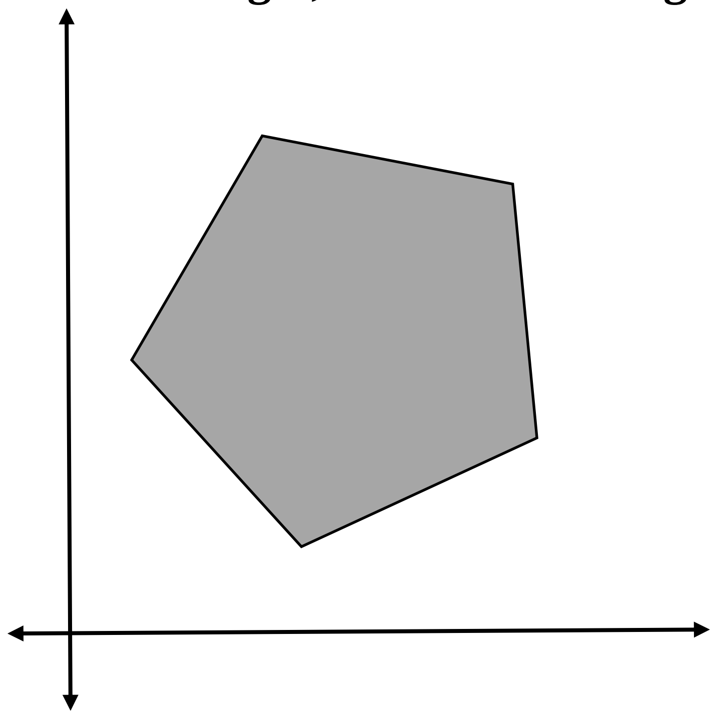
Linear Programming Problems in Two Variables:

A linear programming problem consists of two parts-an objective function and a system of linear inequalities. The objective function is a function of two variables to be maximized/minimized subject to the solutions of the system of linear inequalities. The objective function is usually a linear function of two variables, $z = Ax + By$, that might represent a profit, cost, or revenue. The system of linear inequalities will involve the same two variables as the objective function.

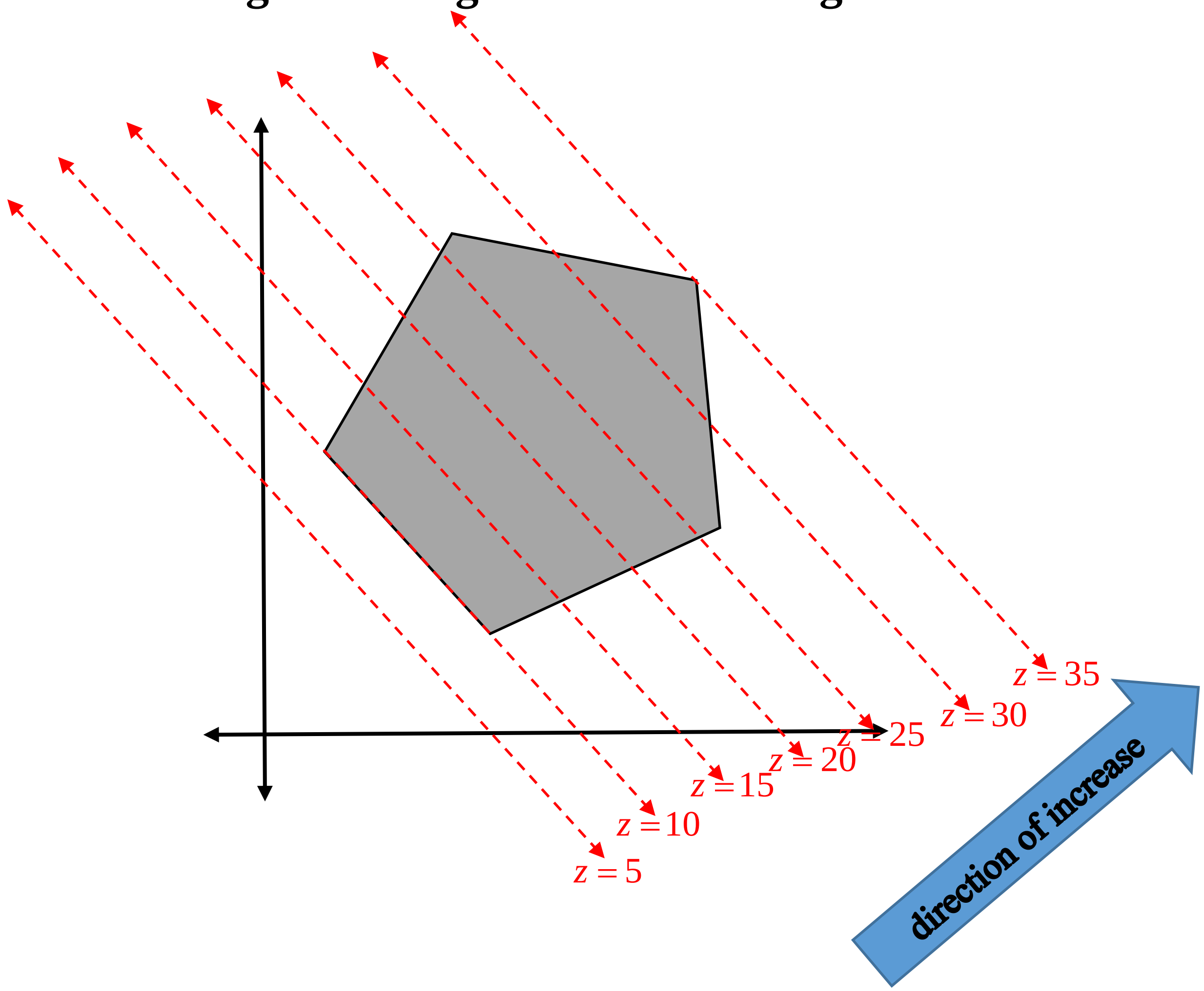
The standard approach to solving a linear programming problem in two variables is to first graph the feasible region determined by the system of linear inequalities. The next step is to determine if there is/are point(s) in the feasible region that give the objective function its largest/smallest value(s). If so, then the linear programming problem has a solution corresponding to such points. If not, then the linear programming problem has no solution.

Closed and Bounded Feasible Region: **If the feasible region is closed(all its edges are solid) and bounded(it can be contained inside a circle), then the objective function will have both a maximum and minimum value.**

In the case of a closed and bounded feasible region, the feasible region will in general be a convex polygon with solid edges, like the following.

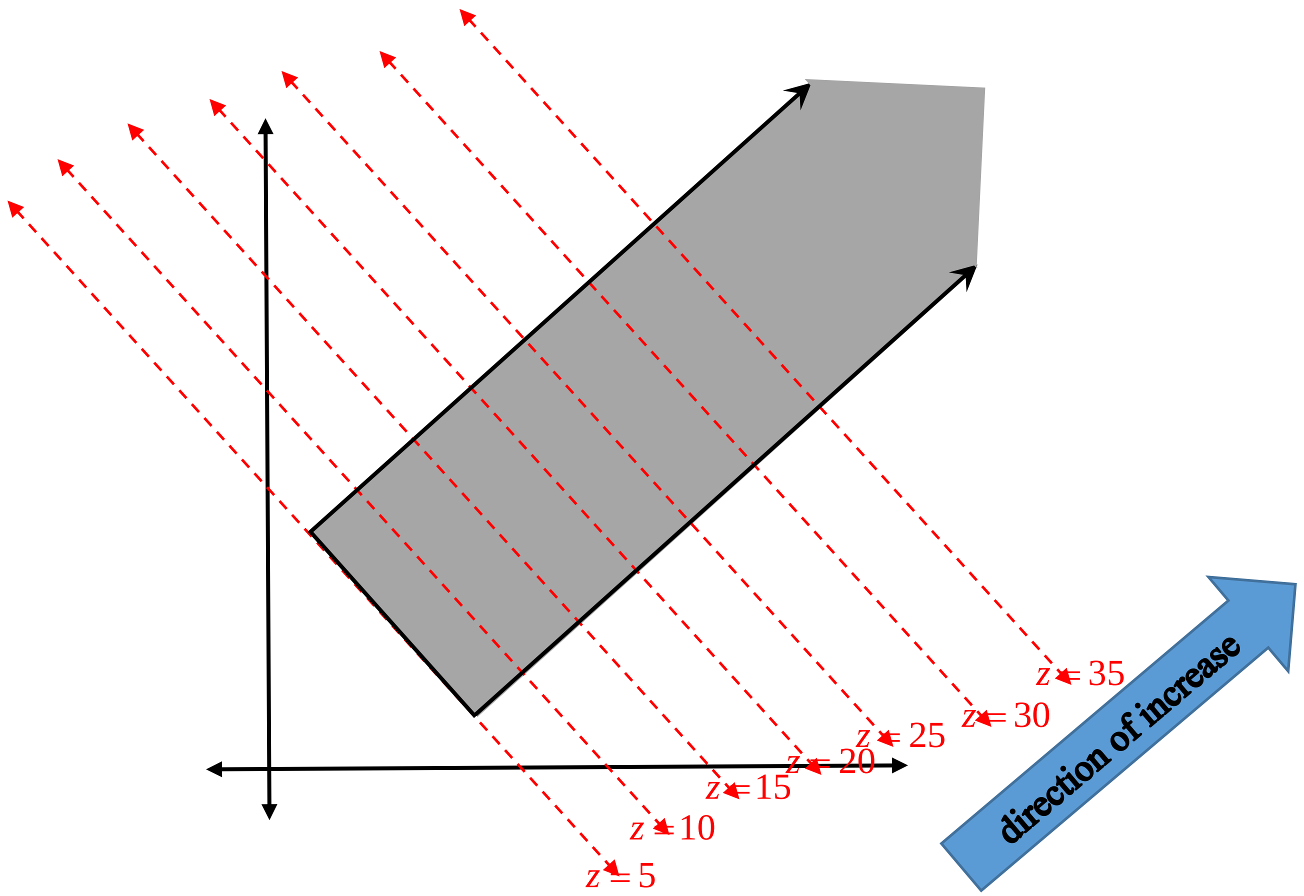


The objective function, $z = Ax + By$, defines what are called iso-objective lines in the *xy*-plane. These are parallel lines each of whose points produce the same value of the objective function. For example, $Ax + By = 5$ would be the line all of whose points produce the value 5 from the objective function. Let's produce a graph of possible iso-objective function lines along with the general feasible region.

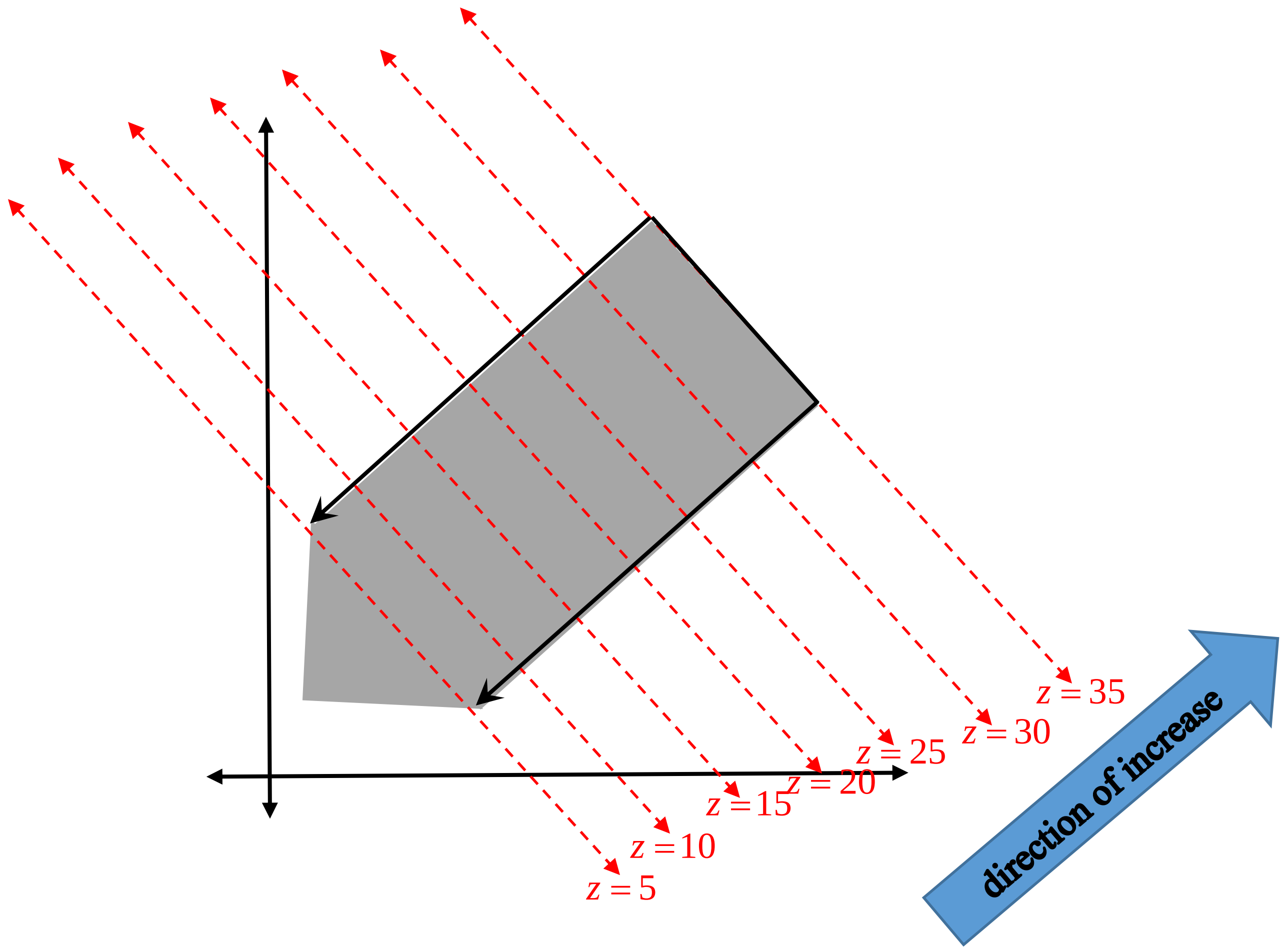


If this is what the graph looks like, then the objective function would have a minimum value of 10 occurring at all the points in the feasible region on the lower left edge and a maximum value of 35 occurring at the corner point in the upper right. Notice that the lower left edge includes two of the corner points, as well. In general, for a closed and bounded feasible region, the objective function will take on a maximum and minimum value and they will occur at corner points.

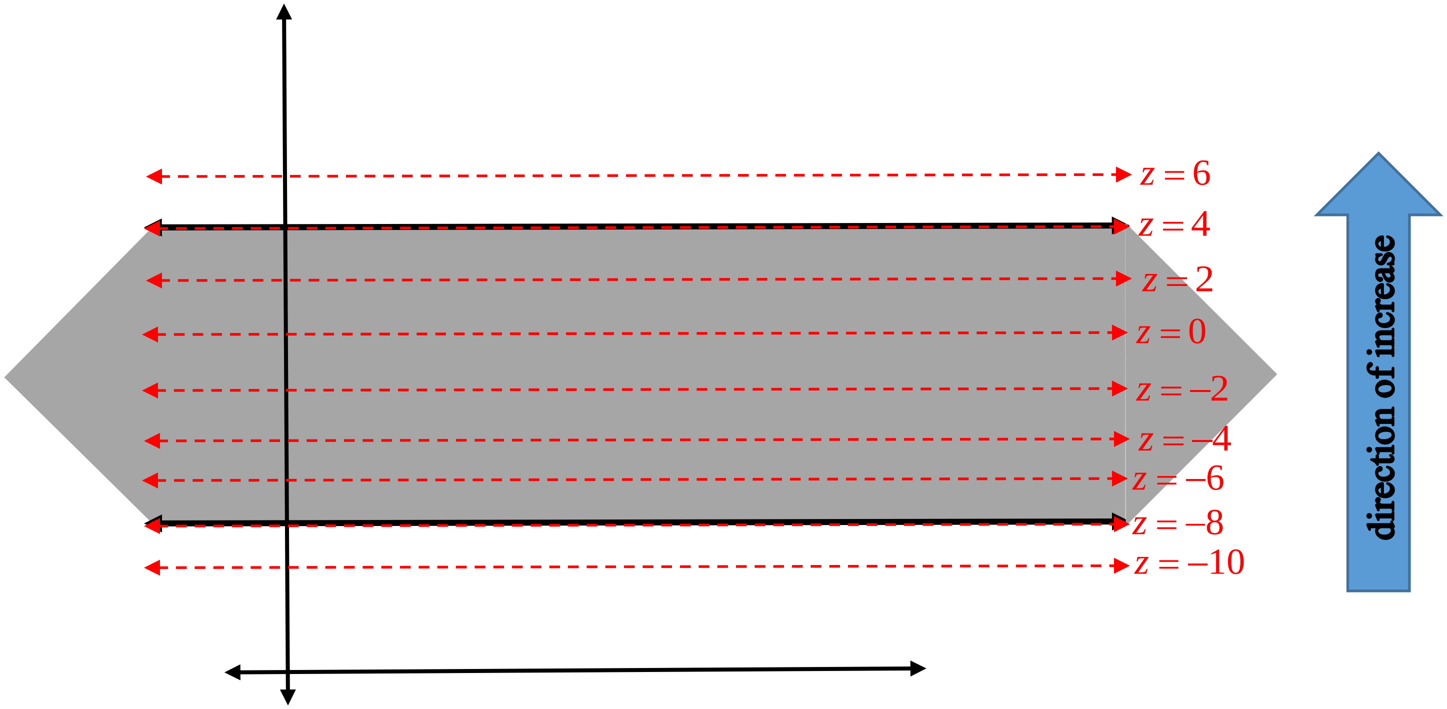
What if the feasible region is closed but unbounded?



If this is what the graph looks like, then the objective function will have a minimum value of 5, but no maximum value. You can see that there are points in this unbounded feasible region that will allow the value of the objective function unlimited growth. The minimum value occurs at corner points.



If this is what the graph looks like, then the objective function will have a maximum value of 35, but no minimum value. You can see that there are points in this unbounded feasible region that will allow the value of the objective function to be unlimited in how small it can get. The maximum value occurs at corner points.

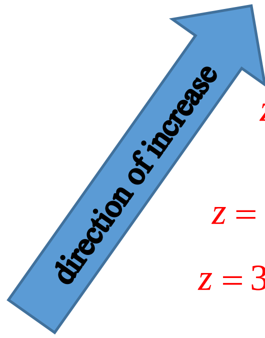


If this is what the graph looks like, then the objective function will have both a maximum and minimum value. The maximum value is 4, and the minimum value is -8. Neither the maximum nor minimum values occur at corner points.

Examples:

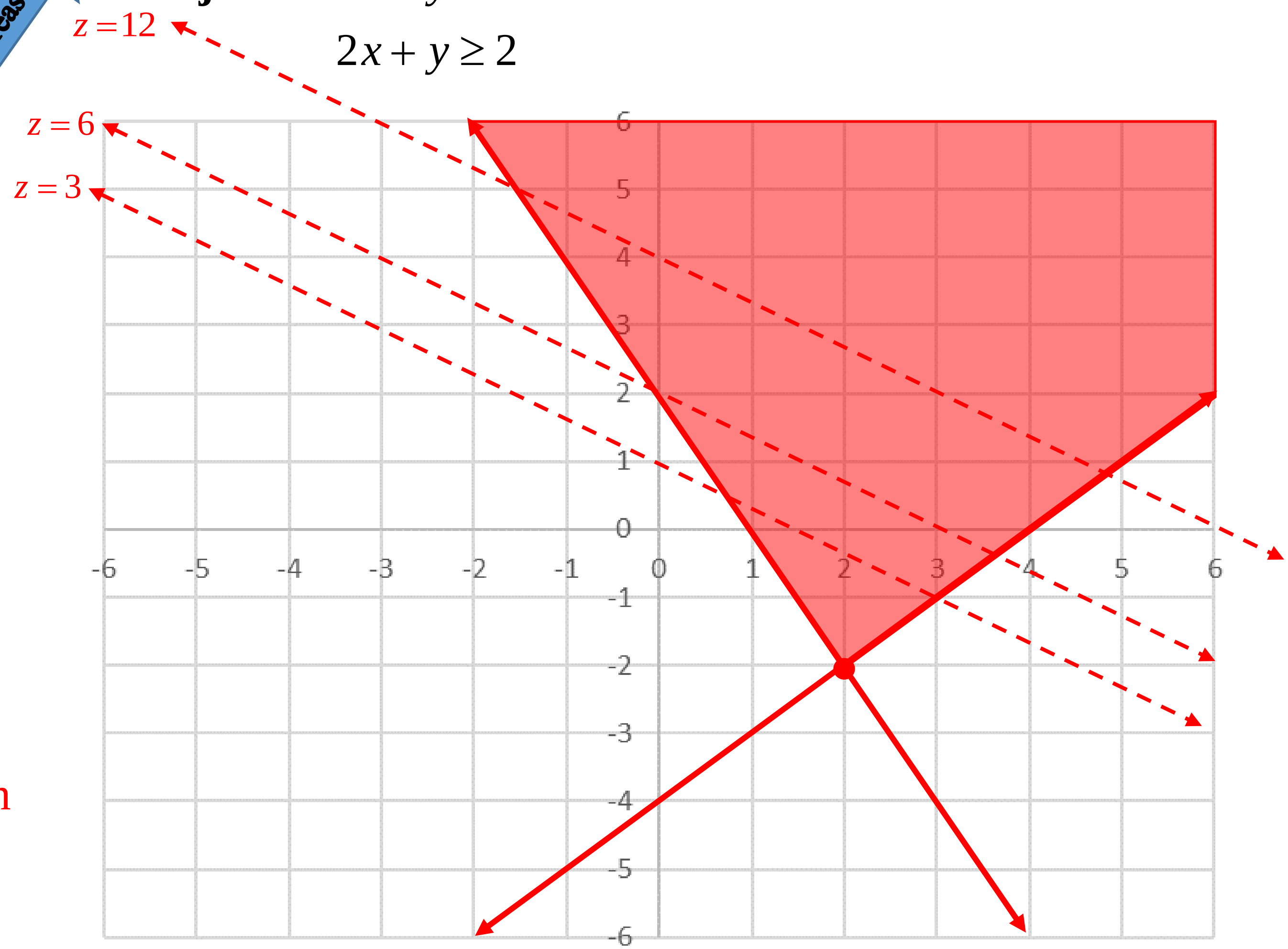
1. Solve the linear programming problem

Maximize and minimize: $z = 2x + 3y$



Subject to $x - y \le 4$

$$2x + y \ge 2$$



Corner Point	$z = 2x + 3y$
$(2, -2)$	-4

The minimum value is -4 and it occurs for $x = 2, y = -2$. There is no maximum value because there is no limit to how large the objective function can get in this unbounded feasible region.

2. Solve the linear programming problem

Maximize and minimize: $z = 2x - 3y$

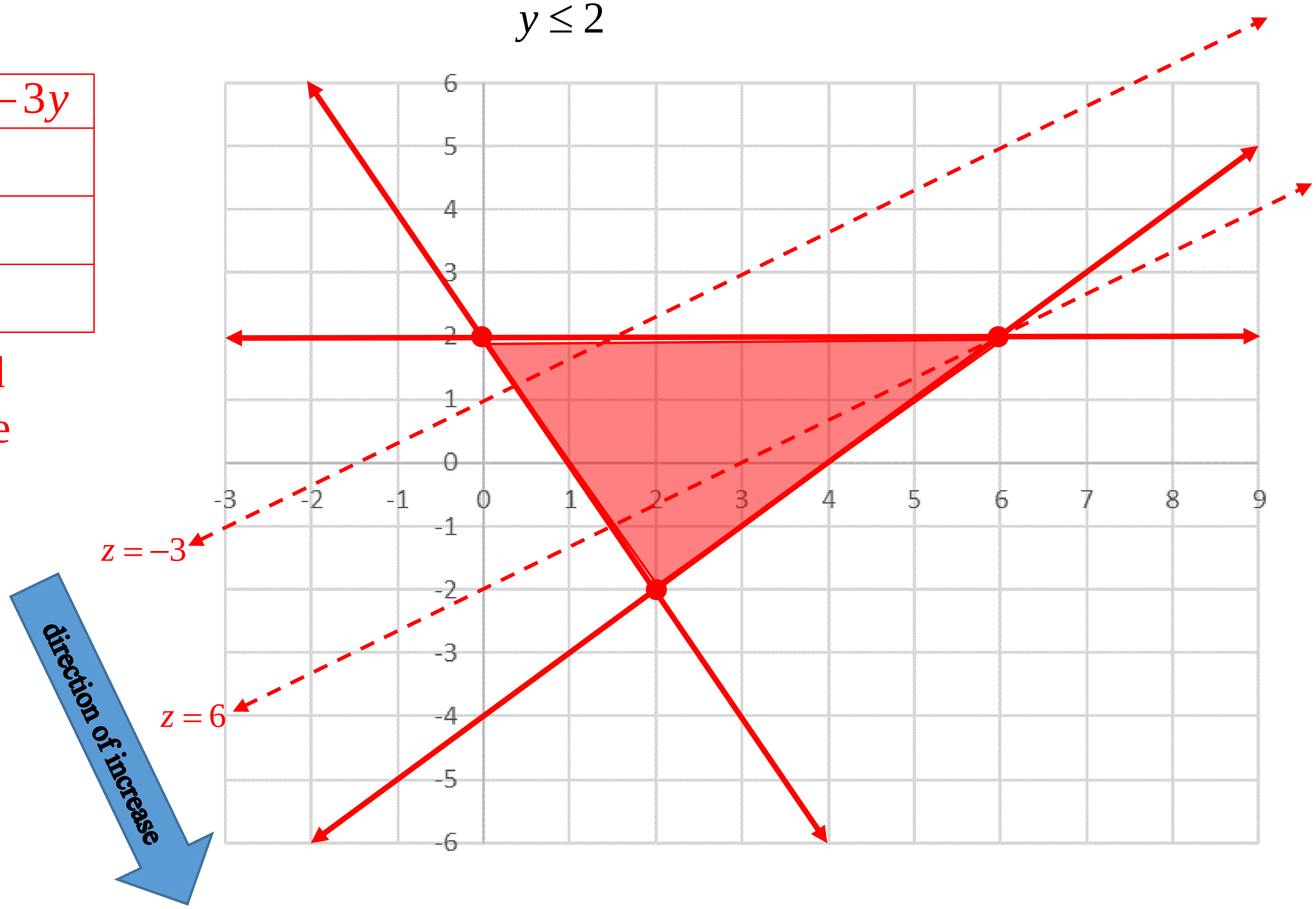
Subject to $x - y \leq 4$

$2x + y \geq 2$

$y \leq 2$

Corner Point	$z = 2x - 3y$
$(2, -2)$	10
$(0, 2)$	-6
$(6, 2)$	6

The minimum value is -6 and it occurs for $x = 0, y = 2$. The maximum value is 10 and it occurs for $x = 2, y = -2$. The feasible region is closed and bounded, so the objective function will have both maximum and minimum values occurring at corner points.



3. Solve the linear programming problem

Maximize and minimize: $z = -x + 2y$

Subject to $2x + 3y \le 12$

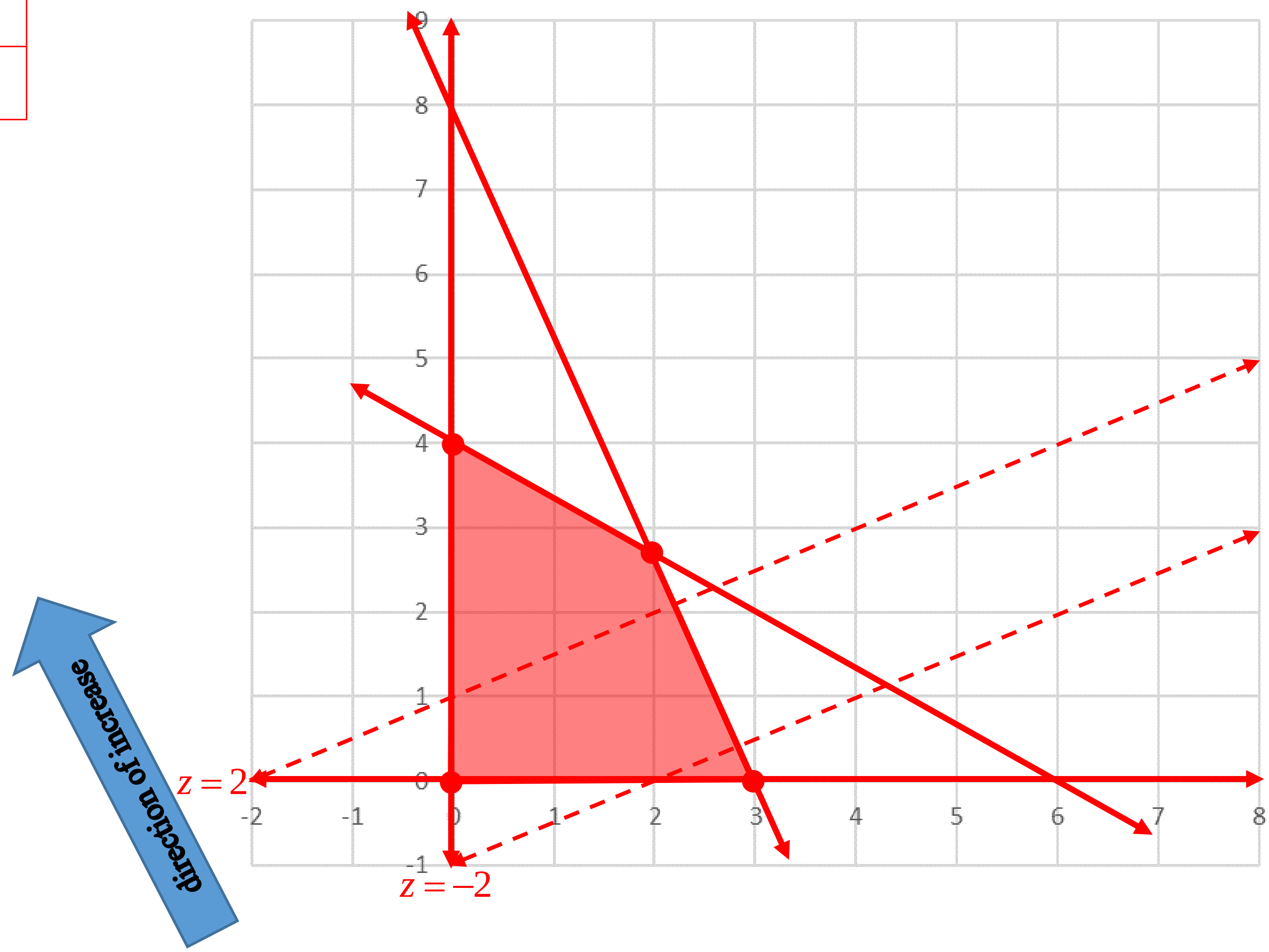
$8x + 3y \le 24$

$x \ge 0$

$y \ge 0$

Corner Point	$z = -x + 2y$
$(0,0)$	0
$(0,4)$	8
$(3,0)$	-3
$\left(2, \frac{8}{3}\right)$	$\frac{10}{3}$

The minimum value is -3 and it occurs for $x = 3, y = 0$. The maximum value is 8 and it occurs for $x = 0, y = 4$. The feasible region is closed and bounded, so the objective function will have both maximum and minimum values occurring at corner points.



4. Solve the linear programming problem

Maximize and minimize: $z = x + 2y$

Subject to $2x + 3y \geq 12$

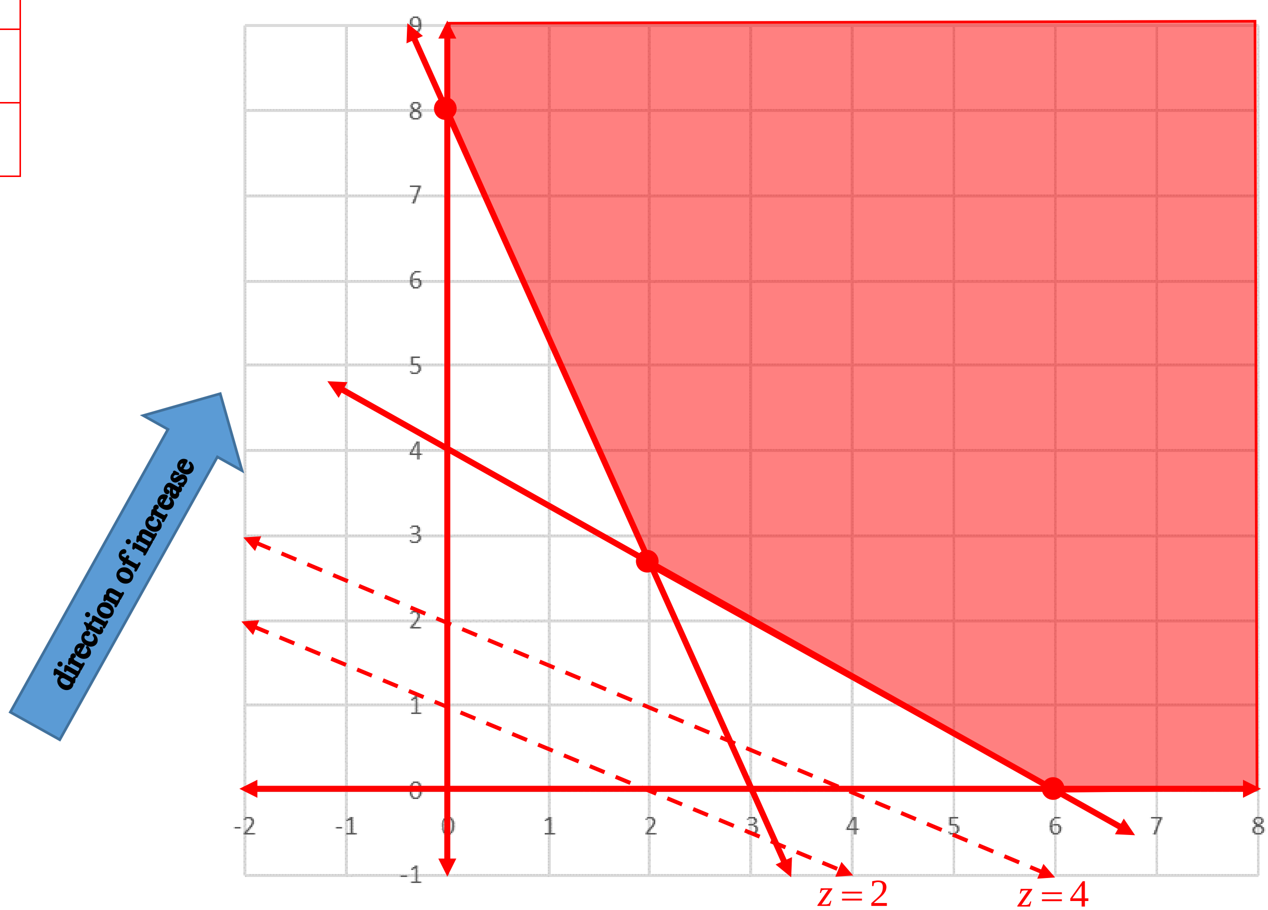
$8x + 3y \geq 24$

$x \geq 0$

$y \geq 0$

Corner Point	$z = x + 2y$
$(0,8)$	16
$\left(2,\frac{8}{3}\right)$	$\frac{22}{3}$
$(6,0)$	6

The minimum value is 6 and it occurs for $x = 6, y = 0$. There is no maximum value because there is no limit to how large the objective function can get in this unbounded feasible region.



Here's an application.

A furniture company makes tables and chairs. The manufacturing information is contained in the following table.

	Table	Chair	Maximum available labor hours
Assembly hours	8	2	400
Finishing hours	2	1	120
Profit per unit	$90	$25	

How many tables and chairs should be made to maximize profit? What is the maximum profit?

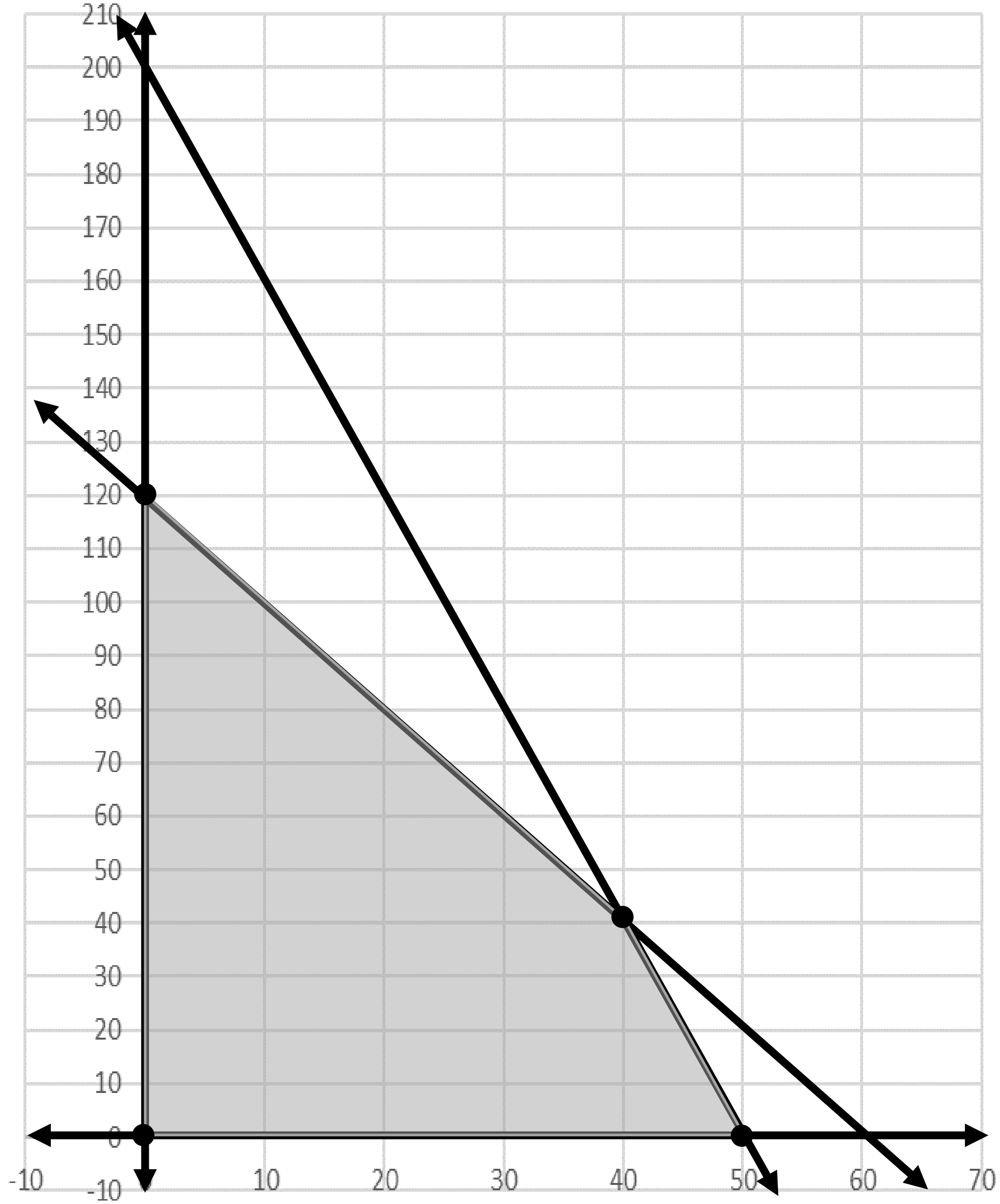
Let x **= # of tables and** y **= # of chairs, and here's the linear programming problem:**

Maximize: $P = 90x + 25y$

Subject to $8x + 2y \le 400$

$2x + y \le 120$

$x, y \ge 0$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70

The feasible region is closed and bounded, so the maximum and minimum values will occur at corner points.

Corner point	$P = 90x + 25y$
$(0,0)$	**0**
$(0,120)$	**3,000**
$(50,0)$	**4,500**
$(40,40)$	**4,600**

If the company makes and sells 40 tables and 40 chairs, they will get their maximum profit of $4,600.