Dealing with unusual events in the Simplex Method:

What if there is a tie for the most negative indicator? You may use either for the selection of the pivot column, but one of the choices might lead to a solution in fewer steps.

Example:

Maximize: $P = x_1 + x_2$

Subject to $2x_1 + x_2 \le 16$

$x_1 \le 6$

$x_2 \le 10$

$x_1, x_2 \ge 0$

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & P \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & 1 & 1 & 0 & 0 & 0 & 16 \\ 1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

There's a tie between the first and second columns for the pivot column. If you choose the first column, it will require three pivots to get to the solution. If you choose the second column, it will only require two pivots. Let's check it out.

First column as pivot column:

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & P & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & 1 & 1 & 0 & 0 & 0 & 16 \\ 1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{-2R_2+R_1\to R_1,\, R_2+R_4\to R_4}{\to} \left[\begin{array}{cccccc|c} 0 & 1 & 1 & -2 & 0 & 0 & 4 \\ 1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline 0 & -1 & 0 & 1 & 0 & 1 & 6 \end{array}\right] \underset{-R_1+R_3\to R_3,\, R_1+R_4\to R_4}{\to} \left[\begin{array}{cccccc|c} 0 & 1 & 1 & -2 & 0 & 0 & 4 \\ 1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & -1 & 2 & 1 & 0 & 6 \\ \hline 0 & 0 & 1 & -1 & 0 & 1 & 10 \end{array}\right] \underset{\frac{1}{2}R_3\to R_3}{\to}$$

$$\left[\begin{array}{cccccc|c} 0 & 1 & 1 & -2 & 0 & 0 & 4 \\ 1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & 0 & 3 \\ \hline 0 & 0 & 1 & -1 & 0 & 1 & 10 \end{array}\right] \underset{2R_3+R_1\to R_1,\, -R_3+R_2\to R_2,\, R_3+R_4\to R_4}{\to} \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 10 \\ 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 3 \\ 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & 0 & 3 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 13 \end{array}\right]$$

So it takes 3 pivots to get the solution of $P=13, x_1=3, x_2=10$, if we start with the first column as the pivot column.

Second column as pivot column:

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & P \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & 1 & 1 & 0 & 0 & 0 & 16 \\ 1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{-R_3+R_1\to R_1,\, R_3+R_4\to R_4}{\rightarrow} \left[\begin{array}{cccccc|c} 2 & 0 & 1 & 0 & -1 & 0 & 6 \\ 1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline -1 & 0 & 0 & 0 & 1 & 1 & 10 \end{array}\right] \underset{\frac{1}{2}R_1\to R_1}{\rightarrow} \left[\begin{array}{cccccc|c} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 3 \\ 1 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline -1 & 0 & 0 & 0 & 1 & 1 & 10 \end{array}\right] \underset{-R_1+R_2\to R_2,\, R_1+R_4\to R_4}{\rightarrow}$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 3 \\ 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 13 \end{array}\right]$$

So it only takes 2 pivots to get the solution of

$P = 13, x_1 = 3, x_2 = 10$, if we start with the second column as the pivot column.

Usually you won't know if one choice is more efficient than the other, so just pick one and go with it.

What if there is a tie for the smallest ratio? You may use either for the selection of the pivot row, however, it is possible, but rare, that you might end up in a repeating cycle of tableaus that don't lead to the solution.

Example:

Maximize: $P = 10x_1 - 57x_2 - 9x_3 - 24x_4$

Subject to $\frac{1}{2}x_1 - \frac{11}{2}x_2 - \frac{5}{2}x_3 + 9x_4 \le 0$

$$\frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + x_4 \le 0$$

$$x_1 \le 1$$

$$x_1, x_2, x_3, x_4 \ge 0$$

$$\left[\begin{array}{cccccccc|c} \frac{1}{2} & -\frac{11}{2} & -\frac{5}{2} & 9 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline -10 & 57 & 9 & 24 & 0 & 0 & 0 & 1 & 0 \end{array}\right]_{2R_1 \to R_1} \rightarrow \left[\begin{array}{cccccccc|c} 1 & -11 & -5 & 18 & 2 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline -10 & 57 & 9 & 24 & 0 & 0 & 0 & 1 & 0 \end{array}\right]_{-\frac{1}{2}R_1 + R_2 \to R_2,\, -R_1 + R_3 \to R_3,\, 10R_1 + R_4 \to R_4} \rightarrow$$

Chose Row 1 over Row 2

$$\left[\begin{array}{cccccccc|c} 1 & -11 & -5 & 18 & 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & -8 & -1 & 1 & 0 & 0 & 0 \\ 0 & 11 & 5 & -18 & -2 & 0 & 1 & 0 & 1 \\ \hline 0 & -53 & -41 & 204 & 20 & 0 & 0 & 1 & 0 \end{array}\right] \underset{\frac{1}{4}R_2 \to R_2}{\to} \left[\begin{array}{cccccccc|c} 1 & -11 & -5 & 18 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -2 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 11 & 5 & -18 & -2 & 0 & 1 & 0 & 1 \\ \hline 0 & -53 & -41 & 204 & 20 & 0 & 0 & 1 & 0 \end{array}\right] \to$$

$$11R_2+R_1 \to R_1, -11R_2+R_3 \to R_3, 53R_2+R_4 \to R_4$$

$$\left[\begin{array}{cccccccc|c} 1 & 0 & \frac{1}{2} & -4 & -\frac{3}{4} & \frac{11}{4} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -2 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 4 & \frac{3}{4} & -\frac{11}{4} & 1 & 0 & 1 \\ \hline 0 & 0 & -\frac{29}{2} & 98 & \frac{27}{4} & \frac{53}{4} & 0 & 1 & 0 \end{array}\right] \underset{2R_1 \to R_1}{\to} \left[\begin{array}{cccccccc|c} 2 & 0 & 1 & -8 & -\frac{3}{2} & \frac{11}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -2 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 4 & \frac{3}{4} & -\frac{11}{4} & 1 & 0 & 1 \\ \hline 0 & 0 & -\frac{29}{2} & 98 & \frac{27}{4} & \frac{53}{4} & 0 & 1 & 0 \end{array}\right] \to$$

$$-\tfrac{1}{2}R_1+R_2 \to R_2, \tfrac{1}{2}R_1+R_3 \to R_3, \tfrac{29}{2}R_1+R_4 \to R_4$$

Chose Row 1 over Row 2

$$\left[\begin{array}{cccccccc|c} 2 & 0 & 1 & -8 & -\frac{3}{2} & \frac{11}{2} & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & \frac{1}{2} & -\frac{5}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 29 & 0 & 0 & -18 & -15 & 93 & 0 & 1 & 0 \end{array}\right] \xrightarrow{} \left[\begin{array}{cccccccc|c} 2 & 0 & 1 & -8 & -\frac{3}{2} & \frac{11}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{4} & -\frac{5}{4} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 29 & 0 & 0 & -18 & -15 & 93 & 0 & 1 & 0 \end{array}\right] \rightarrow$$

$\frac{1}{2}R_2 \rightarrow R_2$ $\qquad$ $8R_2 + R_1 \rightarrow R_1, 18R_2 + R_4 \rightarrow R_4$

$$\left[\begin{array}{cccccccc|c} -2 & 4 & 1 & 0 & \frac{1}{2} & -\frac{9}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{4} & -\frac{5}{4} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 20 & 9 & 0 & 0 & -\frac{21}{2} & \frac{141}{2} & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccccccc|c} -4 & 8 & 2 & 0 & 1 & -9 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{4} & -\frac{5}{4} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 20 & 9 & 0 & 0 & -\frac{21}{2} & \frac{141}{2} & 0 & 1 & 0 \end{array}\right] \rightarrow$$

$2R_1 \rightarrow R_1$ $\qquad$ $-\frac{1}{4}R_1 + R_2 \rightarrow R_2, \frac{21}{2}R_1 + R_4 \rightarrow R_4$

Chose Row 1 over Row 2

$$\left[\begin{array}{cccccccc|c} -4 & 8 & 2 & 0 & 1 & -9 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline -22 & 93 & 21 & 0 & 0 & -24 & 0 & 1 & 0 \end{array}\right] \underset{9R_2+R_1\to R_1,\,24R_2+R_4\to R_4}{\to} \left[\begin{array}{cccccccc|c} \frac{1}{2} & -\frac{11}{2} & -\frac{5}{2} & 9 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline -10 & 57 & 9 & 24 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

All these pivots, and we're right back where we started. If different selections of pivot rows are not made, we'll never make it to the solution: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ **giving** ***P*** **its maximum value of 1.**

What if a unit column is repeated? Unit columns can't be used more than once.

Example:

Maximize: $P = x_1 + x_2$

Subject to $x_1 + x_2 \le 1$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & P \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 2 \\ \hline -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Using either the first or second column as the pivot column results in

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & P \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right].$$

The same unit column appears under both the x_1 **and** x_2 **variables, but we can't use it for both. In this case, the maximum value of *P* is 1 and it occurs at two different corner points:** $x_1 = 1, x_2 = 0$ **and** $x_1 = 0, x_2 = 1$**. In fact, the**

maximum occurs at every point on the segment that joins these two corner points as well. It doesn't, however, occur at the point $x_1 = 1, x_2 = 1$, which isn't even in the feasible region.

Application:

A home builder is planning a new development consisting of three styles of houses: colonial, split-level, and ranch. The details for each style are in the following table.

	Land	Cost	Labor	Profit
colonial	$\frac{1}{2}$ acre	$60,000	4,000 hours	$20,000
split-level	$\frac{1}{2}$ acre	$60,000	3,000 hours	$18,000
ranch	1 acre	$80,000	4,000 hours	$24,000

The builder has 30 acres of land, $3,200,000 in available capital, and 180,000 available labor hours. How many houses of each type should he construct to maximize his profit? What is the maximum profit?

Let x_1 = **# of colonial houses,** x_2 = **# of split-level houses, and** x_3 = **# of ranch houses.**

$$\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 + x_3 \le 30 \text{ (land constraint)}$$
$$60{,}000x_1 + 60{,}000x_2 + 80{,}000x_3 \le 3{,}200{,}000 \text{ (capital constraint)}$$
$$4{,}000x_1 + 3{,}000x_2 + 4{,}000x_3 \le 180{,}000 \text{ (labor hours constraint)}$$
$$x_1, x_2, x_3 \ge 0 \text{ (nonnegative constraint)}$$

Dividing by appropriate powers of 10, it cleans up into

$$\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 + x_3 \le 30$$
$$3x_1 + 3x_2 + 4x_3 \le 160$$
$$4x_1 + 3x_2 + 4x_3 \le 180$$
$$x_1, x_2, x_3 \ge 0$$

So we want to

$$\textbf{Maximize: } P = 20{,}000x_1 + 18{,}000x_2 + 24{,}000x_3$$

$$\textbf{Subject to } \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 + x_3 \le 30$$

$$3x_1 + 3x_2 + 4x_3 \le 160$$

$$4x_1 + 3x_2 + 4x_3 \le 180$$

$$x_1, x_2, x_3 \ge 0$$

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array}$$

$$\left[\begin{array}{ccccccc|c} \frac{1}{2} & \frac{1}{2} & 1 & 1 & 0 & 0 & 0 & 30 \\ 3 & 3 & 4 & 0 & 1 & 0 & 0 & 160 \\ 4 & 3 & 4 & 0 & 0 & 1 & 0 & 180 \\ \hline -20{,}000 & -18{,}000 & -24{,}000 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow$$

$$-4R_1 + R_2 \rightarrow R_2, -4R_1 + R_3 \rightarrow R_3, 24{,}000R_1 + R_4 \rightarrow R_4$$

$$\left[\begin{array}{ccccccc|c} \frac{1}{2} & \frac{1}{2} & 1 & 1 & 0 & 0 & 0 & 30 \\ 1 & 1 & 0 & -4 & 1 & 0 & 0 & 40 \\ 2 & 1 & 0 & -4 & 0 & 1 & 0 & 60 \\ \hline -8{,}000 & -6{,}000 & 0 & 24{,}000 & 0 & 0 & 1 & 720{,}000 \end{array}\right] \rightarrow$$

$$\tfrac{1}{2}R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccccccc|c} \frac{1}{2} & \frac{1}{2} & 1 & 1 & 0 & 0 & 0 & 30 \\ 1 & 1 & 0 & -4 & 1 & 0 & 0 & 40 \\ 1 & \frac{1}{2} & 0 & -2 & 0 & \frac{1}{2} & 0 & 30 \\ \hline -8,000 & -6,000 & 0 & 24,000 & 0 & 0 & 1 & 720,000 \end{array}\right] \rightarrow$$

$$-\tfrac{1}{2}R_3+R_1 \to R_1, -R_3+R_2 \to R_2, 8,000R_3+R_4 \to R_4$$

$$\left[\begin{array}{ccccccc|c} 0 & \frac{1}{4} & 1 & 2 & 0 & -\frac{1}{4} & 0 & 15 \\ 0 & \frac{1}{2} & 0 & -2 & 1 & -\frac{1}{2} & 0 & 10 \\ 1 & \frac{1}{2} & 0 & -2 & 0 & \frac{1}{2} & 0 & 30 \\ \hline 0 & -2,000 & 0 & 8,000 & 0 & 4,000 & 1 & 960,000 \end{array}\right] \rightarrow$$

$$2R_2 \to R_2$$

$$\left[\begin{array}{ccccccc|c} 0 & \frac{1}{4} & 1 & 2 & 0 & -\frac{1}{4} & 0 & 15 \\ 0 & 1 & 0 & -4 & 2 & -1 & 0 & 20 \\ 1 & \frac{1}{2} & 0 & -2 & 0 & \frac{1}{2} & 0 & 30 \\ \hline 0 & -2,000 & 0 & 8,000 & 0 & 4,000 & 1 & 960,000 \end{array}\right] \rightarrow$$

$$-\tfrac{1}{4}R_2+R_1 \to R_1, -\tfrac{1}{2}R_2+R_3 \to R_3, 2,000R_2+R_4 \to R_4$$

$$\left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 3 & -\frac{1}{2} & 0 & 0 & 10 \\ 0 & 1 & 0 & -4 & 2 & -1 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 20 \\ \hline 0 & 0 & 0 & 0 & 4,000 & 2,000 & 1 & 1,000,000 \end{array}\right]$$

So the builder should make 20 colonial houses, 20 split-level houses, and 10 ranch houses to get a maximum profit of $1,000,000.