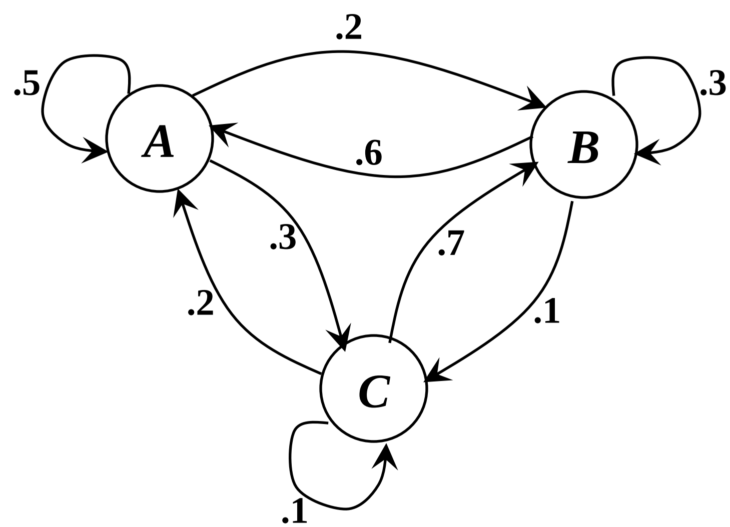
Markov Chains:

A population is divided into disjoint groups called states. Each state has fixed probabilities(proportions) of one of its individuals transitioning into another state or remaining in the same state. These transitions occur at regular time intervals called steps or iterations. Given an initial distribution of the population(probabilities/proportions) among the states, a Markov chain is the sequence of resulting population distributions as the time steps occur. In this way, a Markov chain is a simple model of the future.

Transition Diagram:

The dynamics of a Markov chain are determined by the transition probabilities among the different states of the population. One way to represent these transition probabilities is with a transition diagram. It indicates all of the different states, usually as circles, and all of the transition probabilities from one state to another, usually using arrows.

Example: **Suppose the population consists of the three states *A, B,* and *C*. Here's a possible transition diagram for a Markov chain.**



The sum of the probabilities/proportions emanating from a state/circle must equal 1 since they account for all possibilities. For example, in a given step, the probability that an individual in state *A* will move to state *B* is .2, will move to state *C* is .3, or will remain in state *A* is .5.

Transition Matrix:

Another way to represent the transition probabilities from one state to another is to list them out in matrix form. The current state is always on the left side of the matrix, while the next state is at the top of the matrix. The transition matrix is usually denoted by *P*.

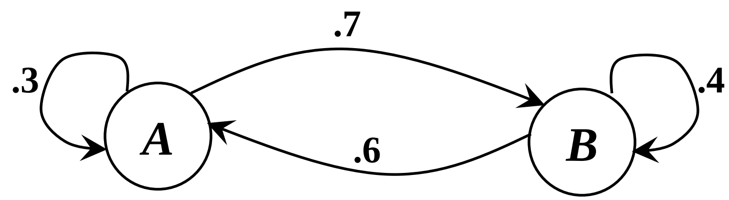
Example: Let's convert the previous transition diagram into a transition matrix.

$$\begin{array}{cc} & \textbf{next state} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \textbf{current state} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} .5 & .2 & .3 \\ .6 & .3 & .1 \\ .2 & .7 & .1 \end{bmatrix} = P \end{array}$$

The fact that the sum of the probabilities emanating from a state must equal 1 translates into the fact the sum of the entries in a row in a transition matrix must equal 1.

Another example:

Convert the given transition diagram into a transition matrix.



$$P = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} .3 & .7 \\ .6 & .4 \end{bmatrix} \end{array}$$

A Markov chain begins by the selection or determination of the initial-state matrix, a row matrix containing the starting probabilities/proportions in each state denoted by S_0. The next state matrix, S_1, is found by multiplying the initial-state matrix, S_0, with the transition matrix, $S_1 = S_0 P$. The next state matrix, similarly, $S_2 = S_1 P$, and so on.

Example:

Suppose that the transition matrix for a Markov chain is $P = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} .1 & .9 \\ .6 & .4 \end{bmatrix} \end{array}$**, and the initial-state matrix is** $S_0 = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} .3 & .7 \end{bmatrix} \end{array}$**.**

Find the first-state matrix, S_1.

$$S_1 = \begin{bmatrix} .3 & .7 \end{bmatrix} \begin{bmatrix} .1 & .9 \\ .6 & .4 \end{bmatrix} = \boxed{\begin{bmatrix} .45 & .55 \end{bmatrix}}$$

Find the second-state matrix, S_2.

$$S_2 = \begin{bmatrix} .45 & .55 \end{bmatrix} \begin{bmatrix} .1 & .9 \\ .6 & .4 \end{bmatrix} = \boxed{\begin{bmatrix} .375 & .625 \end{bmatrix}}$$

<u>Powers of the Transition Matrix:</u>

Given, S_0,

$$\begin{aligned} S_1 &= S_0 P \\ S_2 &= S_1 P = \left(S_0 P\right) P = S_0 P^2 \\ S_3 &= S_2 P = S_0 P^3 \\ S_4 &= S_0 P^4 \\ &\vdots \end{aligned}$$

So in general, $S_k = S_0 P^k$. This means that instead of stepping through the process to determine the k^{th} -state matrix, we can just multiply the initial-state matrix with the k^{th} power of the transition matrix.

Example:

If $P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} .1 & .9 \\ .6 & .4 \end{bmatrix} \end{array}$**, then** $P^2 = \begin{bmatrix} .55 & .45 \\ .3 & .7 \end{bmatrix}$ **and** $P^3 = \begin{bmatrix} .325 & .675 \\ .45 & .55 \end{bmatrix}$**.**

Given $S_0 = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} .3 & .7 \end{bmatrix} \end{array}$**, use the powers of *P* to determine** S_2 **and** S_3**.**

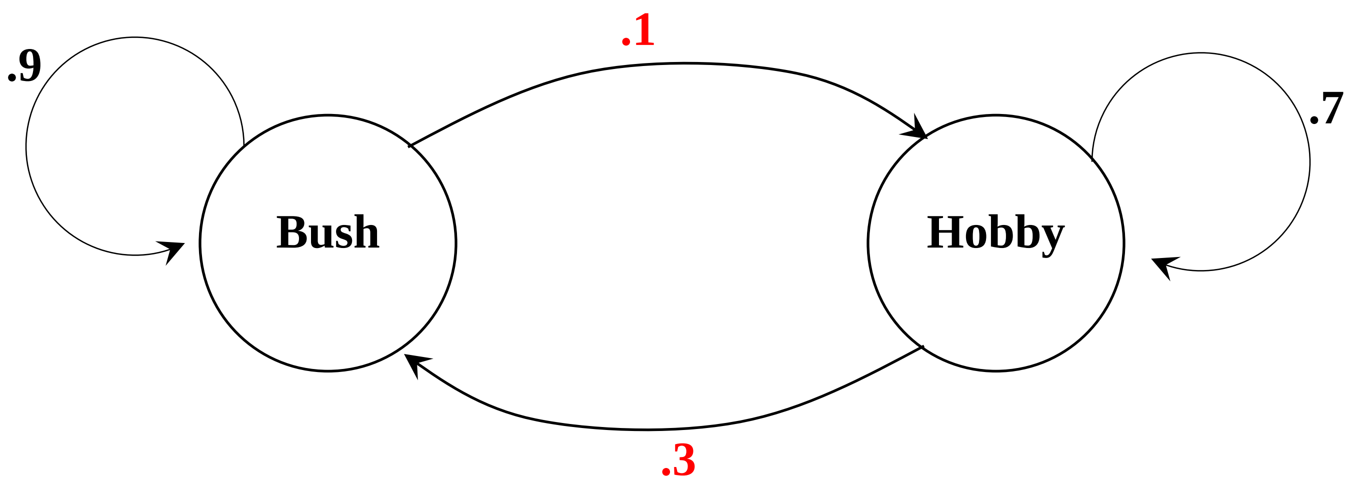
$$S_2 = \begin{bmatrix} .3 & .7 \end{bmatrix} \begin{bmatrix} .55 & .45 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} .375 & .625 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} .3 & .7 \end{bmatrix} \begin{bmatrix} .325 & .675 \\ .45 & .55 \end{bmatrix} = \begin{bmatrix} .4125 & .5875 \end{bmatrix}$$

An Application of Markov Chains:

A car rental agency has offices at Bush and Hobby. Assume that a car rented at either airport must be returned to one of the airports. If a car is rented at Bush, then the probability that it is returned there is .9. If a car is rented at Hobby, then the probability that it is returned there is .7. Assume that the company rents all of its 100 cars each day and that each car is rented and returned only once a day.

Complete the transition diagram:



Complete the transition matrix:

$$\begin{array}{cc} & \begin{array}{cc} B & H \end{array} \\ \begin{array}{c} B \\ H \end{array} & \begin{bmatrix} .9 & .1 \\ .3 & .7 \end{bmatrix} = P \end{array}$$

If the company starts with 50 cars at each airport, then what's the expected distribution on the next day?

This can be done in two ways:

Indirectly using proportions

$$\begin{bmatrix} .5 & .5 \end{bmatrix} \begin{bmatrix} .9 & .1 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} .6 & .4 \end{bmatrix} \Rightarrow \text{60 cars at Bush and 40 cars at Hobby}$$

Or

Directly using numbers

$$\begin{bmatrix} 50 & 50 \end{bmatrix} \begin{bmatrix} .9 & .1 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} 60 & 40 \end{bmatrix} \Rightarrow \text{60 cars at Bush and 40 cars at Hobby}$$

What's the expected/predicted distribution after two days?

$$\begin{bmatrix} 60 & 40 \end{bmatrix} \begin{bmatrix} .9 & .1 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} 66 & 34 \end{bmatrix} \Rightarrow \text{66 cars at Bush and 34 cars at Hobby}$$