Regular Markov Chains:

If all the entries of the transition matrix for a Markov chain are positive or all the entries of a power of the transition matrix are positive, then the associated Markov chain is said to be regular.

Examples:

$$P = \begin{bmatrix} .1 & .9 \\ .3 & .7 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 \\ .3 & .7 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Stationary Matrices for Markov Chains:

The state matrix $S = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}$ **for a Markov chain with transition matrix** P**, is a stationary matrix if** $SP = S$**.**

Examples:

Is $S = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ **a stationary matrix for the Markov chain with transition matrix** $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$**?**

Is $S = \begin{bmatrix} 1 & 0 \end{bmatrix}$ **a stationary matrix for the Markov chain with transition matrix** $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$**?**

Find a stationary matrix for the Markov chain with transition matrix $P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$**?**

Properties of Regular Markov Chains:

If *P* is the transition matrix of a regular Markov chain, then

1) **There is a unique stationary matrix** $S = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}$ **that can be found by solving the system of equations resulting from** $SP = S, s_1 + s_2 + \cdots + s_n = 1$**.**

2) **Given any initial-state matrix** S_0**, the state matrices in the resulting Markov chain,** S_k**'s, will approach the unique stationary matrix *S*, as** $k \to \infty$**.**

3) **The powers of the transition matrix,** P^k**'s, will approach a limiting matrix** $\bar{P}$**, as** $k \to \infty$**, where each row of** $\bar{P}$ **is the unique stationary matrix, *S*.**

So for regular Markov chains, there is a unique distribution of the population so that no matter what the initial-state is, the Markov chain will evolve to this unique distribution as the steps continue.

Examples:

1. Demonstrate the properties for a regular Markov chain with transition matrix,

$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$$\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$$

$$s_1 + s_2 = 1$$

$$\frac{3}{4}s_1 + \frac{1}{2}s_2 = s_1$$

$$\frac{1}{4}s_1 + \frac{1}{2}s_2 = s_2$$

$$s_1 + s_2 = 1$$

$$-\frac{1}{4}s_1 + \frac{1}{2}s_2 = 0$$

$$\frac{1}{4}s_1 - \frac{1}{2}s_2 = 0$$

$$s_1 + s_2 = 1$$

The first two equations are equivalent, so we just have to solve the system

$$\tfrac{1}{4}s_1 - \tfrac{1}{2}s_2 = 0$$
$$s_1 + s_2 = 1$$

Multiplying the first equation by 2 and adding to the second, we get

$\tfrac{3}{2}s_1 = 1 \Rightarrow s_1 = \tfrac{2}{3} \Rightarrow s_2 = \tfrac{1}{3}$**, so the unique stationary matrix is** $S = \begin{bmatrix} \tfrac{2}{3} & \tfrac{1}{3} \end{bmatrix}$**.**

For $S_0 = \begin{bmatrix} \tfrac{1}{4} & \tfrac{3}{4} \end{bmatrix}$**, here's the beginning of the resulting Markov chain:**

S_0	$\begin{bmatrix} \tfrac{1}{4} & \tfrac{3}{4} \end{bmatrix}$
S_1	$\begin{bmatrix} .5625 & .4375 \end{bmatrix}$
S_2	$\begin{bmatrix} .640625 & .359375 \end{bmatrix}$
S_3	$\begin{bmatrix} .660156 & .339844 \end{bmatrix}$
S_4	$\begin{bmatrix} .665039 & .334961 \end{bmatrix}$
S_5	$\begin{bmatrix} .666260 & .333740 \end{bmatrix}$

You can see the Markov chain heading toward the stationary matrix $\begin{bmatrix} \tfrac{2}{3} & \tfrac{1}{3} \end{bmatrix}$**.**

For $S_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, here's the beginning of the resulting Markov chain:

S_0	$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
S_1	$\begin{bmatrix} .625 & .375 \end{bmatrix}$
S_2	$\begin{bmatrix} .65625 & .34375 \end{bmatrix}$
S_3	$\begin{bmatrix} .664063 & .335938 \end{bmatrix}$
S_4	$\begin{bmatrix} .666016 & .333984 \end{bmatrix}$
S_5	$\begin{bmatrix} .666504 & .333496 \end{bmatrix}$

You can see the Markov chain heading toward the stationary matrix $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

For $S_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, here's the beginning of the resulting Markov chain:

S_0	$\begin{bmatrix} 1 & 0 \end{bmatrix}$
S_1	$\begin{bmatrix} .75 & .25 \end{bmatrix}$
S_2	$\begin{bmatrix} .6875 & .3125 \end{bmatrix}$
S_3	$\begin{bmatrix} .671875 & .328125 \end{bmatrix}$
S_4	$\begin{bmatrix} .667969 & .332031 \end{bmatrix}$
S_5	$\begin{bmatrix} .666992 & .333008 \end{bmatrix}$

You can see the Markov chain heading toward the stationary matrix $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

Here are the powers of $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$**:**

$$P^2 = \begin{bmatrix} .6875 & .3125 \\ .625 & .375 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} .671875 & .328125 \\ .65625 & .34375 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} .667969 & .332031 \\ .664063 & .335938 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} .666992 & .333008 \\ .666016 & .333984 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} .666748 & .333252 \\ .666504 & .333496 \end{bmatrix}$$

$$P^7 = \begin{bmatrix} .666687 & .333313 \\ .666626 & .333374 \end{bmatrix}$$

You can see the powers of the transition matrix heading toward the limiting matrix $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$**.**

2. Find the stationary matrix and the limiting matrix $\bar{P}$ for the Markov chain with transition matrix, $P=\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$. Determine the long-term behavior of the Markov chain with initial-state of $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$$\begin{bmatrix} s_1 & s_2 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}=\begin{bmatrix} s_1 & s_2 \end{bmatrix}$$

$$s_1+s_2=1$$

$$\tfrac{3}{4}s_1+\tfrac{2}{3}s_2=s_1$$

$$\tfrac{1}{4}s_1+\tfrac{1}{3}s_2=s_2$$

$$s_1+s_2=1$$

$$-\tfrac{1}{4}s_1+\tfrac{2}{3}s_2=0$$

$$\tfrac{1}{4}s_1-\tfrac{2}{3}s_2=0$$

$$s_1+s_2=1$$

3. Find the stationary matrix and the limiting matrix $\bar{P}$ for the Markov chain with transition matrix, $P = \begin{bmatrix} .5 & .1 & .4 \\ .3 & .7 & 0 \\ 0 & .6 & .4 \end{bmatrix}$. Determine the long-term behavior of the Markov chain with initial-state of $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} .5 & .1 & .4 \\ .3 & .7 & 0 \\ 0 & .6 & .4 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$$

$$s_1 + s_2 + s_3 = 1$$

$$s_1 + s_2 + s_3 = 1$$

$$-.5s_1 + .3s_2 = 0$$

$$.1s_1 - .3s_2 + .6s_3 = 0$$

$$.4s_1 - .6s_3 = 0$$

$$\Rightarrow s_2 = \tfrac{5}{3}s_1, s_3 = \tfrac{2}{3}s_1 \Rightarrow s_1 + \tfrac{5}{3}s_1 + \tfrac{2}{3}s_1 = 1 \Rightarrow$$

For the Markov chain with transition matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is a stationary matrix,

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. **Notice that**

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**,…, which means that the powers of the transition matrix don't settle on a particular matrix. Also notice that if** $S_0 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}$**, then** $S_1 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}$**,** $S_2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}$**,…, which means that this Markov Chain doesn't evolve toward the stationary matrix** $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$**.**

Do these observations contradict the properties of a regular Markov chain?

<u>Application of Regular Markov Chains:</u>

A new rapid transit system has just started operating. In the first month of operation, it is found that 25% of commuters use the system, while 75% still travel by car. The following transition matrix was determined by examining data from other systems.

$$\begin{matrix} & R & C \\ R & .8 & .2 \\ C & .3 & .7 \end{matrix} \quad \begin{matrix} R \\ C \end{matrix}\begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} = P$$

, where *R* represents using rapid transit and *C* represents using a car.

The time step in this example is 1 month.

1) What's the initial-state matrix?

2) What percentage of commuters will be using rapid transit after 1 month?

$$S_1 = \begin{bmatrix} .25 & .75 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} =$$

3) What percentage of commuters will be using rapid transit after 2 months?

$$S_2 = \begin{bmatrix} .425 & .575 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} =$$

4) Find the percentage of commuters using each type of transportation after the new system has been in service for a long time.

$$\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$$

$$s_1 + s_2 = 1$$

$$.8s_1 + .3s_2 = s_1$$
$$.2s_1 + .7s_2 = s_2$$
$$s_1 + s_2 = 1$$

$$-.2s_1 + .3s_2 = 0$$
$$.2s_1 - .3s_2 = 0$$
$$s_1 + s_2 = 1$$