Regular Markov Chains:

If all the entries of the transition matrix for a Markov chain are positive or all the entries of a power of the transition matrix are positive, then the associated Markov chain is said to be regular.

Examples:

$$P = \begin{bmatrix} .1 & .9 \\ .3 & .7 \end{bmatrix}$$

All the entries are positive, so the associated Markov chain is regular.

$$P = \begin{bmatrix} 0 & 1 \\ .3 & .7 \end{bmatrix}$$

$P^2 = \begin{bmatrix} 0 & 1 \\ .3 & .7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} .3 & .7 \\ .21 & .79 \end{bmatrix}$, since all the entries of P^2 are positive, the associated Markov chain is regular.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{7}{8} \\ 0 & 1 \end{bmatrix}$$

$$\vdots$$

$$P^k = \begin{bmatrix} \frac{1}{2^k} & \frac{2^k-1}{2^k} \\ 0 & 1 \end{bmatrix}$$

Since no power of P will ever have all positive entries, the associated Markov chain is not regular.

Stationary Matrices for Markov Chains:

The state matrix $S = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}$ **for a Markov chain with transition matrix** P**, is a stationary matrix if** $SP = S$**.**

Examples:

Is $S = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ **a stationary matrix for the Markov chain with transition matrix** $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$**?**

$$SP = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} = S$$

So yes, $S = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ is a stationary matrix.

Is $S = \begin{bmatrix} 1 & 0 \end{bmatrix}$ a stationary matrix for the Markov chain with transition matrix $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$?

$$SP = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \neq S$$

So no, $S = \begin{bmatrix} 1 & 0 \end{bmatrix}$ is not a stationary matrix.

Find a stationary matrix for the Markov chain with transition matrix $P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$?

We need to find $S = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$ with $\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$, $s_1 + s_2 = 1$.

This leads to the system of equations $\begin{matrix} -\frac{3}{4}s_1 + \frac{3}{4}s_2 = 0 \\ \frac{3}{4}s_1 - \frac{3}{4}s_2 = 0 \\ s_1 + s_2 = 1 \end{matrix}$, which reduces to the system $\begin{matrix} s_1 - s_2 = 0 \\ s_1 + s_2 = 1 \end{matrix}$. So $s_1 = s_2 = \frac{1}{2}$, and the stationary matrix is $S = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Properties of Regular Markov Chains:

If *P* is the transition matrix of a regular Markov chain, then

1) There is a unique stationary matrix $S = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}$ **that can be found by solving the system of equations resulting from** $SP = S, s_1 + s_2 + \cdots + s_n = 1$**.**

2) Given any initial-state matrix S_0**, the state matrices in the resulting Markov chain,** S_k **'s, will approach the unique stationary matrix *S*, as** $k \to \infty$**.**

3) The powers of the transition matrix, P^k **'s, will approach a limiting matrix** $\bar{P}$**, as** $k \to \infty$**, where each row of** $\bar{P}$ **is the unique stationary matrix, *S*.**

So for regular Markov chains, there is a unique distribution of the population so that no matter what the initial-state is, the Markov chain will evolve to this unique distribution as the steps continue. Although the proportions in the states stabilize over time, the individuals in the population continue to move among the states indefinitely!

Examples:

1. Demonstrate the properties for a Markov chain with transition matrix, $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

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$$\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$$

$$s_1 + s_2 = 1$$

$$\tfrac{3}{4} s_1 + \tfrac{1}{2} s_2 = s_1$$
$$\tfrac{1}{4} s_1 + \tfrac{1}{2} s_2 = s_2$$
$$s_1 + s_2 = 1$$

$$-\tfrac{1}{4} s_1 + \tfrac{1}{2} s_2 = 0$$
$$\tfrac{1}{4} s_1 - \tfrac{1}{2} s_2 = 0$$
$$s_1 + s_2 = 1$$

The first two equations are equivalent, so we just have to solve the system

$$\tfrac{1}{4}s_1 - \tfrac{1}{2}s_2 = 0$$
$$s_1 + s_2 = 1$$

Multiplying the first equation by 2 and adding to the second, we get

$\tfrac{3}{2}s_1 = 1 \Rightarrow s_1 = \tfrac{2}{3} \Rightarrow s_2 = \tfrac{1}{3}$**, so the unique stationary matrix is** $S = \begin{bmatrix} \tfrac{2}{3} & \tfrac{1}{3} \end{bmatrix}$**.**

For $S_0 = \begin{bmatrix} \tfrac{1}{4} & \tfrac{3}{4} \end{bmatrix}$**, here's the beginning of the resulting Markov chain:**

S_0	$\begin{bmatrix} \tfrac{1}{4} & \tfrac{3}{4} \end{bmatrix}$
S_1	$\begin{bmatrix} .5625 & .4375 \end{bmatrix}$
S_2	$\begin{bmatrix} .640625 & .359375 \end{bmatrix}$
S_3	$\begin{bmatrix} .660156 & .339844 \end{bmatrix}$
S_4	$\begin{bmatrix} .665039 & .334961 \end{bmatrix}$
S_5	$\begin{bmatrix} .666260 & .333740 \end{bmatrix}$

You can see the Markov chain heading toward the stationary matrix $\begin{bmatrix} \tfrac{2}{3} & \tfrac{1}{3} \end{bmatrix}$**.**

For $S_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, here's the beginning of the resulting Markov chain:

S_0	$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
S_1	$\begin{bmatrix} .625 & .375 \end{bmatrix}$
S_2	$\begin{bmatrix} .65625 & .34375 \end{bmatrix}$
S_3	$\begin{bmatrix} .664063 & .335938 \end{bmatrix}$
S_4	$\begin{bmatrix} .666016 & .333984 \end{bmatrix}$
S_5	$\begin{bmatrix} .666504 & .333496 \end{bmatrix}$

You can see the Markov chain heading toward the stationary matrix $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

For $S_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, here's the beginning of the resulting Markov chain:

S_0	$\begin{bmatrix} 1 & 0 \end{bmatrix}$
S_1	$\begin{bmatrix} .75 & .25 \end{bmatrix}$
S_2	$\begin{bmatrix} .6875 & .3125 \end{bmatrix}$
S_3	$\begin{bmatrix} .671875 & .328125 \end{bmatrix}$
S_4	$\begin{bmatrix} .667969 & .332031 \end{bmatrix}$
S_5	$\begin{bmatrix} .666992 & .333008 \end{bmatrix}$

You can see the Markov chain heading toward the stationary matrix $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

Here are the powers of $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$:

$$P^2 = \begin{bmatrix} .6875 & .3125 \\ .625 & .375 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} .671875 & .328125 \\ .65625 & .34375 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} .667969 & .332031 \\ .664063 & .335938 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} .666992 & .333008 \\ .666016 & .333984 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} .666748 & .333252 \\ .666504 & .333496 \end{bmatrix}$$

$$P^7 = \begin{bmatrix} .666687 & .333313 \\ .666626 & .333374 \end{bmatrix}$$

You can see the powers of the transition matrix heading toward the limiting matrix $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$**.**

2. Find the stationary matrix and the limiting matrix $\bar{P}$ for the Markov chain with transition matrix, $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$. Determine the long-term behavior of the Markov chain with initial-state of $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$$\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$$

$$s_1 + s_2 = 1$$

$$s_1 + s_2 = 1$$

$$-\tfrac{1}{4}s_1 + \tfrac{2}{3}s_2 = 0$$

$$\tfrac{1}{4}s_1 - \tfrac{2}{3}s_2 = 0$$

$$-\tfrac{1}{4}(s_1 + s_2 = 1)$$

$$\underline{+\tfrac{1}{4}s_1 - \tfrac{2}{3}s_2 = 0}$$

$$-\tfrac{11}{12}s_2 = -\tfrac{1}{4}$$

$$s_2 = \tfrac{3}{11}, s_1 = \tfrac{8}{11}$$

So the stationary matrix is $S = \begin{bmatrix} \frac{8}{11} & \frac{3}{11} \end{bmatrix}$.

And the limiting matrix is $\bar{P} = \begin{bmatrix} \frac{8}{11} & \frac{3}{11} \\ \frac{8}{11} & \frac{3}{11} \end{bmatrix}$.

With an initial-state of $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, the state matrices of the Markov chain will approach the stationary matrix $S = \begin{bmatrix} \frac{8}{11} & \frac{3}{11} \end{bmatrix}$.

3. Find the stationary matrix and the limiting matrix $\bar{P}$ for the Markov chain with transition matrix, $P = \begin{bmatrix} .5 & .1 & .4 \\ .3 & .7 & 0 \\ 0 & .6 & .4 \end{bmatrix}$. Determine the long-term behavior of the Markov chain with initial-state of $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} .5 & .1 & .4 \\ .3 & .7 & 0 \\ 0 & .6 & .4 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$$

$$s_1 + s_2 + s_3 = 1$$

$$s_1 + s_2 + s_3 = 1$$
$$-.5s_1 + .3s_2 = 0$$
$$.1s_1 - .3s_2 + .6s_3 = 0$$
$$.4s_1 - .6s_3 = 0$$

$$\Rightarrow s_2 = \tfrac{5}{3}s_1, s_3 = \tfrac{2}{3}s_1 \Rightarrow s_1 + \tfrac{5}{3}s_1 + \tfrac{2}{3}s_1 = 1 \Rightarrow \tfrac{10}{3}s_1 = 1$$

$$\Rightarrow s_1 = \tfrac{3}{10}, s_2 = \tfrac{1}{2}, s_3 = \tfrac{1}{5}$$

so, and the stationary matrix is $S = \begin{bmatrix} \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}$.

And the limiting matrix is $\bar{P} = \begin{bmatrix} \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}$.

With an initial-state of $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$, the state matrices of the Markov chain will approach the stationary matrix $S = \begin{bmatrix} \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}$.

For the Markov chain with transition matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is a stationary matrix, $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Notice that

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,…, which means that the powers of the transition matrix don't settle on a particular matrix. Also notice that if $S_0 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}$, then $S_1 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}$, $S_2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}$,…, which means that this Markov Chain doesn't evolve toward the stationary matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Do these observations contradict the properties of a regular Markov chain?

No, for the transition matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,…, and no power of P has all positive entries, so it's not regular!

Application of Regular Markov Chains:

A new rapid transit system has just started operating. In the first month of operation, it is found that 25% of commuters use the system, while 75% still travel by car. The following transition matrix was determined by examining data from other systems.

$$\begin{array}{c} \\ R \\ C \end{array}\begin{array}{c} \begin{array}{cc} R & C \end{array} \\ \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} \end{array} = P$$

, where *R* represents using rapid transit and *C* represents using a car.

The time step in this example is 1 month.

1) What's the initial-state matrix?

$$S_0 = \begin{bmatrix} .25 & .75 \end{bmatrix}$$

2) What percentage of commuters will be using rapid transit after 1 month?

$$S_1 = \begin{bmatrix} .25 & .75 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} .425 & .575 \end{bmatrix}$$

So the percentage of commuters using rapid transit is 42.5%.

3) What percentage of commuters will be using rapid transit after 2 months?

$$S_2 = \begin{bmatrix} .425 & .575 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} .5125 & .4875 \end{bmatrix}$$

So the percentage of commuters using rapid transit is 51.25%.

4) Find the percentage of commuters using each type of transportation after the new system has been in service for a long time.

$$\begin{bmatrix} s_1 & s_2 \end{bmatrix}\begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$$

$$s_1 + s_2 = 1$$

$$s_1 + s_2 = 1$$

$$-.2s_1 + .3s_2 = 0$$

$$.2s_1 - .3s_2 = 0$$

$$.3(s_1 + s_2 = 1)$$

$$\underline{+.2s_1 - .3s_2 = 0}$$

$$.5s_1 = .3$$

$$s_1 = \tfrac{3}{5}, s_2 = \tfrac{2}{5}$$

So 60% of commuters will be using rapid transit, and 40% of commuters will travel by car.