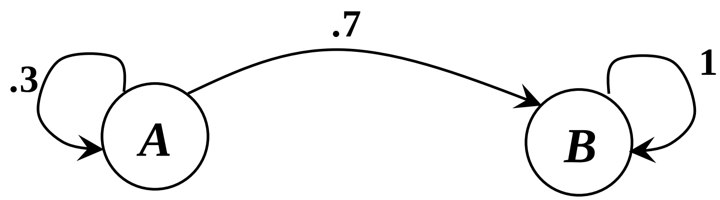
<u>Absorbing Markov Chains:</u>

Absorbing State: **A state in a Markov chain is called an absorbing state, if once it's entered, it's impossible to leave.**

Examples:

1.



2.

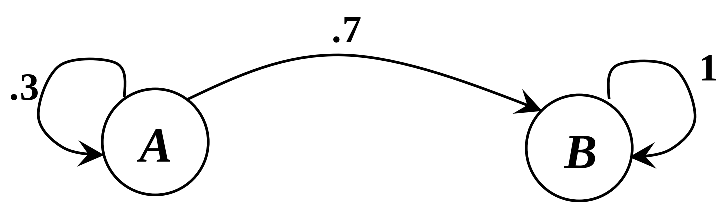
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & .4 & .1 \end{bmatrix} \end{array} = P$$

Absorbing states in a transition diagram have no arrows pointing away from them. In a transition matrix, a row with a 1 on the diagonal and zeros everywhere else indicates an absorbing state.

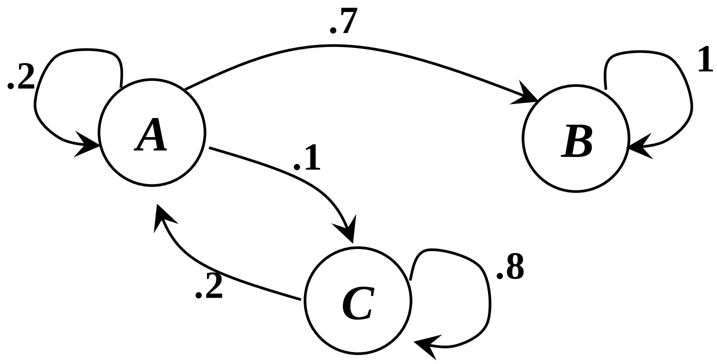
A Markov chain is an *absorbing Markov chain* if there is at least one absorbing state, and it's possible to go from each non-absorbing state to at least one of the absorbing states.

Examples:

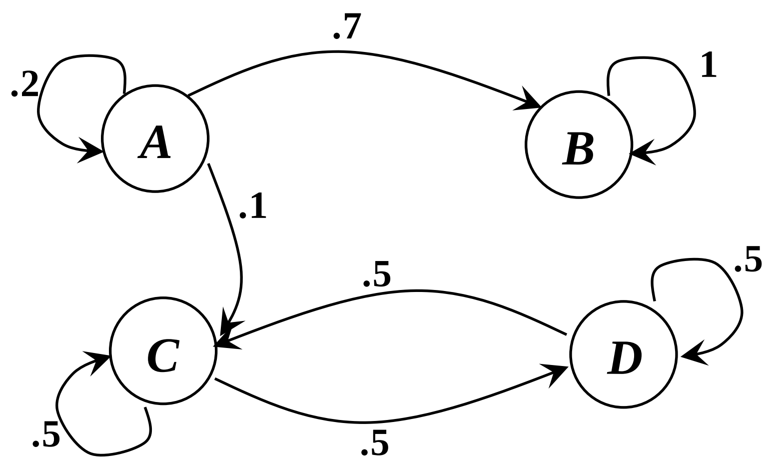
1.



2.



3.



4.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & .4 & .1 \end{bmatrix} \end{array} = P$$

5.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = P$$

Standard Form for an Absorbing Markov Chain:

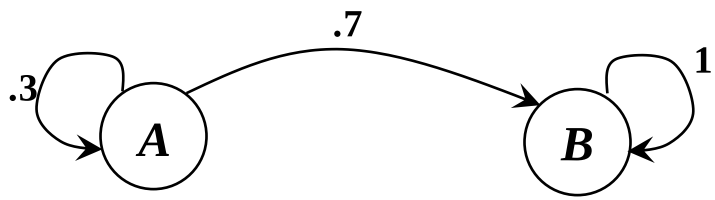
The transition matrix for an absorbing Markov chain is in standard form if the absorbing states precede the non-absorbing states, and the matrix can be partitioned into the form

$$\begin{array}{c} \\ A \\ N \end{array} \begin{array}{c} \begin{array}{cc} A & N \end{array} \\ \left[\begin{array}{c|c} I & O \\ \hline R & Q \end{array}\right] \end{array} = P$$

, where *A* represents the absorbing states, *N* represents the non-absorbing states, *I* is an identity matrix, and *O* is a zero matrix.

Examples: **Find standard form transition matrices for the following absorbing Markov chains. Find S_1 from the given initial-state matrix.**

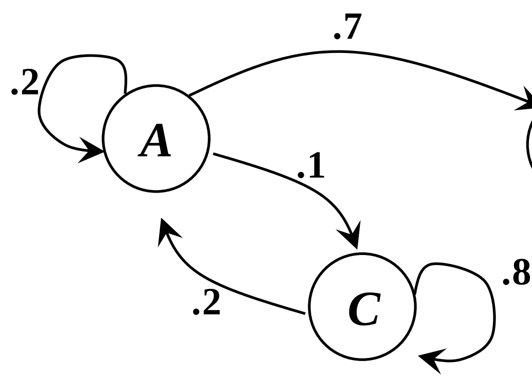
1.



$$\begin{array}{c} \\ B \\ A \end{array}\begin{array}{c} \begin{array}{cc} B & A \end{array} \\ \left[\begin{array}{c|c} 1 & 0 \\ \hline .7 & .3 \end{array}\right] \end{array} = P$$

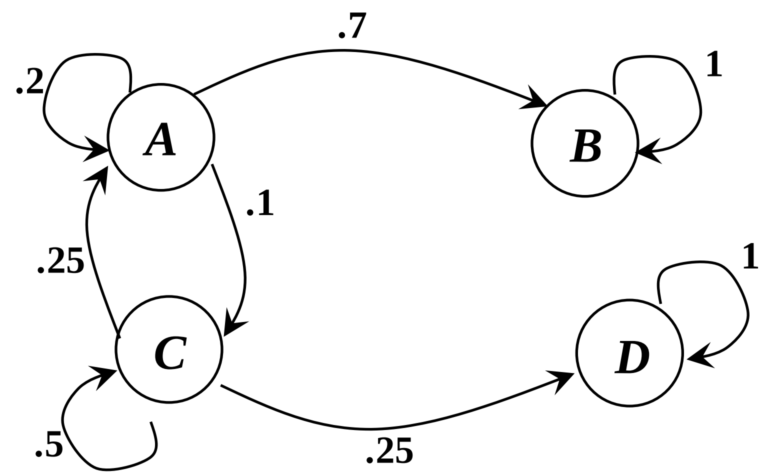
, the identity matrix, *I*, is the number 1, and the zero matrix, *O*, is just the number 0. If $S_0 = \overset{\begin{array}{cc} A & B \end{array}}{\begin{bmatrix} .2 & .8 \end{bmatrix}}$**, then we first have to change the order of the states to get** $S_0 = \overset{\begin{array}{cc} B & A \end{array}}{\begin{bmatrix} .8 & .2 \end{bmatrix}}$**, and then multiply** $S_1 = S_0 P = \begin{bmatrix} .8 & .2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ .7 & .3 \end{bmatrix} = \overset{\begin{array}{cc} B & A \end{array}}{\begin{bmatrix} .94 & .06 \end{bmatrix}}$**.**

2.



$$\begin{matrix} & A & B & C \\ S_0 = & [.1 & .3 & .6] \end{matrix}$$

3.



$$\begin{matrix} & A & B & C & D \\ S_0 = & [.1 & .1 & .6 & .2] \end{matrix}$$

For absorbing Markov chains, as the process continues, members of the population begin to accumulate in the absorbing states. As the process continues indefinitely, all members of the population end up in an absorbing state. We are interested in the probabilities of ending up in particular absorbing states. For regular Markov chains, the limiting population distribution is independent of the initial-state matrix. This is not the case for an absorbing Markov chain - where you end up depends on where you start.

I start going and end up doing, finish everything, and conclude nothing? What am I?

<u>The Limiting Matrix for an Absorbing Markov Chain:</u>

If a standard form transition matrix *P* for an absorbing Markov chain is partitioned as

$$\begin{array}{cc} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \left[\begin{array}{c|c} I & O \\ \hline R & Q \end{array}\right] \end{array},$$

then P^k approaches a limiting matrix, $\bar{P}$, as *k* increases, where

$$\bar{P} = \left[\begin{array}{c|c} I & O \\ \hline FR & O \end{array}\right].$$

The matrix $F = (I - Q)^{-1}$ is called the *<u>fundamental matrix</u>* for *P*. The identity matrix used in calculating *F* has the same dimension as the matrix *Q*.

Examples: Convert the transition matrices to a standard form and find the limiting matrix. Determine the long-term distribution from the two given initial-states.

1. $$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ .3 & .4 & .3 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = P$$

$$\begin{array}{c} \\ A \\ C \\ B \end{array}\begin{array}{c} \begin{array}{ccc} A & C & B \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline .3 & .3 & .4 \end{array}\right] \end{array} = P$$

$R = [.3 \quad .3]$ **and** $Q = [.4]$, **so** $F = (I - Q)^{-1} = (1 - .4)^{-1} = (.6)^{-1} = \frac{5}{3}$ **and**

$FR = \frac{5}{3}[.3 \quad .3] = [.5 \quad .5]$**, so the limiting matrix is** $$\begin{array}{c} \\ A \\ C \\ B \end{array}\begin{array}{c} \begin{array}{ccc} A & C & B \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline .5 & .5 & 0 \end{array}\right] \end{array} = \overline{P}$$**. This means that**

if you start in B and the chain continues indefinitely, the long-term probability of ending up in A is .5 and the long-term probability of ending up in C is .5.

If $S_0 = \begin{matrix} A & B & C \\ [.1 & .2 & .7] \end{matrix}$**, we'll first have to rearrange the states,** $S_0 = \begin{matrix} A & C & B \\ [.1 & .7 & .2] \end{matrix}$**, before we multiply.** $[.1 \quad .7 \quad .2]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & .5 & 0 \end{bmatrix} = [.2 \quad .8 \quad 0]$**, so the initial-state** $\begin{matrix} A & C & B \\ [.1 & .7 & .2] \end{matrix}$ **leads to a long-term distribution of** $\begin{matrix} A & C & B \\ [.2 & .8 & 0] \end{matrix}$**.**

If $S_0 = \begin{matrix} A & B & C \\ [.3 & .5 & .2] \end{matrix}$**, we'll first have to rearrange the states,** $S_0 = \begin{matrix} A & C & B \\ [.3 & .2 & .5] \end{matrix}$**, before we multiply.** $[.3 \quad .2 \quad .5]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & .5 & 0 \end{bmatrix} = [.55 \quad .45 \quad 0]$**, so the initial-state** $\begin{matrix} A & C & B \\ [.3 & .2 & .5] \end{matrix}$ **leads to a long-term distribution of** $\begin{matrix} A & C & B \\ [.55 & .45 & 0] \end{matrix}$.

2.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} .3 & 0 & .2 & .5 \\ .4 & .6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = P$$

$$\begin{array}{cccc} A & B & C & D \end{array} \qquad\qquad \begin{array}{cccc} A & B & C & D \end{array}$$

$$S_0 = \begin{bmatrix} .6 & .1 & .1 & .2 \end{bmatrix} \textbf{ and } S_0 = \begin{bmatrix} .4 & .2 & .3 & .1 \end{bmatrix}$$

More about the Limiting Matrix, $\overline{P}$***:***

The sum of the entries in each row of the matrix $F=(I-Q)^{-1}$**, is the expected number of trials it will take to go from each non-absorbing state to some absorbing state.**

Examples:

1. For an absorbing Markov chain with transition matrix $\begin{matrix} & A & B \\ A & \frac{1}{2} & \frac{1}{2} \\ B & 0 & 1 \end{matrix}$**, we get the standard form transition matrix** $\begin{matrix} & B & A \\ B & 1 & 0 \\ A & \frac{1}{2} & \frac{1}{2} \end{matrix}$**.** $F=(I-Q)^{-1}=\left(1-\frac{1}{2}\right)^{-1}=\left(\frac{1}{2}\right)^{-1}=2$**, so the expected number of trials for a member of** A **to be absorbed into** B **is 2.**

Let's verify this. If *X* is the number of trials to go from *A* to *B*, then

X	**1**	**2**	**3**	**4**	**5**	**6**	**…**
P(X)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	**…**

$$
\begin{aligned}
E(X) &= 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + 5\cdot\frac{1}{32} + 6\cdot\frac{1}{64} + \cdots \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \\
&\quad + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \\
&\quad\quad + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \\
&\quad\quad\quad + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \\
&\quad\quad\quad\quad + \frac{1}{32} + \frac{1}{64} + \cdots \\
&\quad\quad\quad\quad\quad \vdots \quad \cdots
\end{aligned}
$$

Suppose that $T=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots$**, then**

$T=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\right)$**, and this means that**

$T=\frac{1}{2}+\frac{1}{2}T\Rightarrow\frac{1}{2}T=\frac{1}{2}\Rightarrow T=1.$ **So** $E(X)=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=2$**, and we've verified it!**

2. For an absorbing Markov chain with transition matrix $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} .3 & 0 & .2 & .5 \\ .4 & .6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$**, find the expected number of trials to go from each non-absorbing state to an absorbing state.**

Remember: $F = (I - Q)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .3 & 0 \\ .4 & .6 \end{bmatrix} \right)^{-1} = \begin{bmatrix} .7 & 0 \\ -.4 & .4 \end{bmatrix}^{-1} = \begin{array}{c} A \\ B \end{array} \begin{bmatrix} \frac{10}{7} & 0 \\ \frac{10}{7} & \frac{5}{2} \end{bmatrix}$

An Application of Absorbing Markov Chains:

Once a year, company employees are given the opportunity to join one of three pension plans: *A*, *B*, or *C*. Once an employee decides to join one of these plans, the employee can't drop or switch to another plan. Past records indicate that each year 4% of employees join plan A, 14% join plan B, 7% join plan C, and 75% don't join any of the plans.

1. **In the long-run, what percentages of employees will choose to join plan *A*, plan *B*, and plan *C*?**

Let's start with a standard form transition matrix:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline .04 & .14 & .07 & .75 \end{array}\right] \end{array} = P$$

$F=(1-.75)^{-1}=(.25)^{-1}=4$, **so** $FR=4[.04 \quad .14 \quad .07]=[.16 \quad .56 \quad .28]$, **and leads to a limiting matrix of**

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline .16 & .58 & .28 & 0 \end{array}\right] \end{array} = \bar{P}$$

. So in the long-run, 16% will choose plan ***A*****, 58% will choose plan** ***B*****, and 28% will choose plan** ***C*****.**

2. What's the expected number of years for an employee to choose one of the three plans?

$F=(1-.75)^{-1}=(.25)^{-1}=4$**, so the expected number of years is 4.**