<u>Leontief Input-Output Analysis for 3 Industries and Unusual Happenings:</u>

Three-Industry Example:

An economy is based on three industrial sectors: agriculture(*A*), building(*B*), and energy(*E*). Each dollar's worth of agriculture output requires an input of 30 cents from agriculture, 10 cents from building, and 30 cents from energy. Each dollar's worth of output from building requires 20 cents from agriculture, 10 cents from building, and 20 cents from energy. Each dollar's worth of energy output requires 10 cents from agriculture, 10 cents from building, and 10 cents from energy.

Complete the technology matrix, *M*.

$$\begin{array}{c} \\ A \\ B \\ E \end{array} \begin{array}{c} \begin{array}{ccc} A & B & E \end{array} \\ \begin{bmatrix} .3 & .2 & .1 \\ .1 & .1 & .1 \\ .3 & .2 & .1 \end{bmatrix} \end{array} = M$$

Suppose that the final demands from the rest of the economy are $20 million worth of agriculture, $5 million worth of building, and $10 million worth of energy. Complete the final demand matrix, *D*, in millions of dollars.

$$\begin{bmatrix} 20 \\ 5 \\ 10 \end{bmatrix} = D$$

In matrix form, the Leontief model of this economy is $X = MX + D$**, which can be rearranged into** $X - MX = D$ **and eventually into** $(I-M)X = D$**. If** $(I-M)^{-1}$ **exists, then we can solve for the outputs from the three sectors by performing the matrix multiplication,** $X = (I-M)^{-1}D$**. In this example,** $I - M = \begin{bmatrix} .7 & -.2 & -.1 \\ -.1 & .9 & -.1 \\ -.3 & -.2 & .9 \end{bmatrix}$

and $\left(I-M\right)^{-1}=\begin{bmatrix}1.58 & .4 & .22\\ .24 & 1.2 & .16\\ .58 & .4 & 1.22\end{bmatrix}$**. Solve for the outputs by completing the following matrix multiplication.**

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}1.58 & .4 & .22\\ .24 & 1.2 & .16\\ .58 & .4 & 1.22\end{bmatrix}\begin{bmatrix}20\\ 5\\ 10\end{bmatrix}=\begin{bmatrix}35.8\\ 12.4\\ 25.8\end{bmatrix}$$

Agriculture output:

$35.8 million

Building output:

$12.4 million

Energy output:

$25.8 million

If the final demands change to $15 million of agriculture, $10 million of building, and $20 million of energy, then find the new output levels for the three sectors.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.58 & .4 & .22 \\ .24 & 1.2 & .16 \\ .58 & .4 & 1.22 \end{bmatrix} \begin{bmatrix} 15 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 32.1 \\ 18.8 \\ 37.1 \end{bmatrix}$$

Agriculture output: $32.1 million

Building output: $18.8 million

Energy output: $37.1 million

If the final demands change to $10 million of agriculture, $8 million of building, and $30 million of energy, then find the new output levels for the three sectors.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.58 & .4 & .22 \\ .24 & 1.2 & .16 \\ .58 & .4 & 1.22 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 30 \end{bmatrix} = \begin{bmatrix} 25.6 \\ 16.8 \\ 45.6 \end{bmatrix}$$

Agriculture output: $25.6 million

Building output: $16.8 million

Energy output: $45.6 million

What happens if $(I-M)^{-1}$ doesn't exist?

Here's an example:

A two-industry economy has the industrial sectors: agriculture(A) and energy(E). Each dollar of agriculture output requires 70 cents of agriculture and 30 cents of energy. Each dollar of energy output requires 80 cents of agriculture and 20 cents of energy. The technology matrix is $M=\begin{bmatrix} .7 & .8 \\ .3 & .2 \end{bmatrix}$ and $I-M=\begin{bmatrix} .3 & -.8 \\ -.3 & .8 \end{bmatrix}$. In this case, $(I-M)^{-1}$ doesn't exist. First let's see if the two industries can support themselves $(d_1, d_2 = 0)$ using row operations.

$$\left[\begin{array}{cc|c} .3 & -.8 & 0 \\ -.3 & .8 & 0 \end{array}\right] \underset{R_1+R_2\to R_2}{\to} \left[\begin{array}{cc|c} .3 & -.8 & 0 \\ 0 & 0 & 0 \end{array}\right] \underset{\frac{1}{.3}R_1\to R_1}{\to} \left[\begin{array}{cc|c} 1 & -\frac{8}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_2 = t, x_1 = \tfrac{8}{3}t; t \ge 0$$

So as long as the output from agriculture is $\frac{8}{3}$ times the output from energy, then they can support each other. What happens if general final demands from the rest of the economy are included?

$$\left[\begin{array}{cc|c} .3 & -.8 & d_1 \\ -.3 & .8 & d_2 \end{array}\right] \underset{R_1+R_2\to R_2}{\rightarrow} \left[\begin{array}{cc|c} .3 & -.8 & d_1 \\ 0 & 0 & d_1+d_2 \end{array}\right]_{\frac{1}{.3}R_1\to R_1}$$

In order for there to be a solution, the $d_1 + d_2$ in the second row would have to equal 0. Since the rest of the economy are consumers and not producers, $d_1, d_2 \geq 0$. This means that the only way that $d_1 + d_2 = 0$ is to have both of them equal to zero. So in this economy, the two industries can support themselves in a non-unique way, but can't support the rest of the economy. *Notice in this example that there is a balance between the two sectors-a dollar's worth of output from each sector requires a total of a dollar's worth input from the two sectors. This means that there's none left for others.*

An out of balance example:

A two-industry economy has the industrial sectors: energy(*E*) and water(*W*). Each dollar of energy output requires 20 cents of energy and 90 cents of water. Each dollar of water output requires 80 cents of energy and 20 cents of water. The technology matrix is $M = \begin{bmatrix} .2 & .8 \\ .9 & .2 \end{bmatrix}$ **and** $I - M = \begin{bmatrix} .8 & -.8 \\ -.9 & .8 \end{bmatrix}$**. Let's see what happens when we use** $(I-M)^{-1}$ **to solve a problem with general demands.** $(I-M)^{-1} = \begin{bmatrix} -10 & -10 \\ -11.25 & -10 \end{bmatrix}$**, so**

$\begin{bmatrix} -10 & -10 \\ -11.25 & -10 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -10d_1 - 10d_2 \\ -11.25d_1 - 10d_2 \end{bmatrix}$**, and this would mean that the outputs from the two sectors would have to be less than or equal to zero. In this example, each dollar's worth of energy requires \$1.10 from the two sectors, while each dollar's worth of water requires a total of a dollar's worth from the two sectors. Two dollars of output requires \$2.10 of input, so the two sectors can't even support themselves.**