

Gauss-Jordan Elimination:

There is an extension of Gaussian Elimination called Gauss-Jordan Elimination.

In general, the goal is to use row operations to reach a matrix with the following form:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & \alpha \\ 0 & 1 & 0 & \cdots & 0 & \beta \\ 0 & 0 & 1 & 0 & \vdots & \gamma \\ \vdots & \vdots & 0 & 1 & 0 & \delta \\ \vdots & \vdots & \vdots & \vdots & & \varepsilon \\ 0 & 0 & 0 & 0 & & \phi \end{array} \right]$$

There are as many 1's as possible on the diagonal with zeros both below the 1's and above the 1's.

Examples:

1. $2x_1 + 3x_2 = -1$
 $3x_1 - 4x_2 = 7$

$$\left[\begin{array}{cc|c} 2 & 3 & -1 \\ 3 & -4 & 7 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 3 & -4 & 7 \\ 2 & 3 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 2 & 3 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 17 & -17 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 1 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array} \right]$$

$R_1 \leftrightarrow R_2$ $-R_2 + R_1 \rightarrow R_1$ $-2R_1 + R_2 \rightarrow R_2$ $\frac{1}{17}R_2 \rightarrow R_2$ $7R_2 + R_1 \rightarrow R_1$

Now that the goal has been reached, you can easily see that the only solution is $x_1 = 1$ and $x_2 = -1$.

$$\begin{array}{l} 2. \quad x_1 + x_2 = 1 \\ \quad -2x_1 - 2x_2 = 2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & -2 & 2 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 4 \end{array} \right]$$

$$2R_1 + R_2 \rightarrow R_2$$

Now that the goal has been reached, convert the last row back into an equation:

$$0 = 4.$$

Since this is impossible, the system has no solution. If at any time in the process of reaching the goal, you get a row with zeros to the left of the bar and a non-zero number to the right, you may stop and conclude that the system has no solution.

$$\begin{array}{l} 3. \quad x_1 + x_2 = 1 \\ \quad 3x_1 + 3x_2 = 3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 3 & 3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

$$-3R_1 + R_2 \rightarrow R_2$$

Now that the goal has been reached, convert the last row back into an equation:

$$0 = 0.$$

There is no contradiction, and we can't uniquely solve for the values of the variables from the first row(equation): $x_1 + x_2 = 1$.

When this happens, the system has infinitely many solutions, and we represent them as follows:

Let $x_2 = t$, and substitute this into the first equation to get $x_1 + t = 1 \Rightarrow x_1 = 1 - t$.

The solutions of the system are given by $x_1 = 1 - t, x_2 = t$; where t is any real number .

$$\begin{aligned} 4. \quad & x_1 - 2x_2 + x_3 = 2 \\ & -3x_1 + x_2 + 2x_3 = 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -3 & 1 & 2 & 4 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -5 & 5 & 10 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -1 & -2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{array} \right]$$

$$3R_1 + R_2 \rightarrow R_2 \qquad -\frac{1}{5}R_2 \rightarrow R_2 \qquad 2R_2 + R_1 \rightarrow R_1$$

Now that the goal has been reached, notice that there are no contradictions, and we can't uniquely solve for the values of the variables. The system has infinitely many solutions, and we'll represent them by setting the variable furthest to the right equal to t : $x_3 = t$.

Substitute this into the last row(equation) to get $x_2 - x_3 = -2 \Rightarrow x_2 - t = -2 \Rightarrow x_2 = t - 2$.

Substitute the x_3 into the first row(equation) to get

$$x_1 - x_3 = -2 \Rightarrow x_1 - t = -2 \Rightarrow x_1 = t - 2.$$

The solutions of the system are given by

$$x_1 = t - 2, x_2 = t - 2, x_3 = t; \text{ where } t \text{ is any real number.}$$

More Examples:

1.
$$\begin{aligned}x_1 - 2x_2 &= 1 \\ 2x_1 - x_2 &= 5\end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -1 & 5 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array} \right]$$

$-2R_1 + R_2 \rightarrow R_2 \quad \frac{1}{3}R_2 \rightarrow R_2 \quad 2R_2 + R_1 \rightarrow R_1$

So we can easily see that the only solution of this system is $x_1 = 3$ and $x_2 = 1$.

$$\begin{array}{l} 2. \quad x_1 + 2x_2 = 4 \\ \quad \quad 2x_1 + 4x_2 = -8 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & -8 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & -16 \end{array} \right]$$
$$-2R_1 + R_2 \rightarrow R_2$$

Since the last row is equivalent to $0 = -16$, we know that this system has no solution.

$$\begin{aligned} 3. \quad & 3x_1 - 6x_2 = -9 \\ & -2x_1 + 4x_2 = 6 \end{aligned}$$

$$\left[\begin{array}{cc|c} 3 & -6 & -9 \\ -2 & 4 & 6 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -3 \\ -2 & 4 & 6 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 0 & 0 \end{array} \right]$$

$$\frac{1}{3}R_1 \rightarrow R_1 \qquad 2R_1 + R_2 \rightarrow R_2$$

We've gone as far as possible, there are no contradictions, and we can't uniquely solve for the variables. This means that there are infinitely many solutions. Starting with the variable on the right, let $x_2 = t$, and the first row turns into $x_1 - 2t = -3 \Rightarrow x_1 = 2t - 3$. So the solutions are $x_1 = 2t - 3$ and $x_2 = t$; where t is any real number.

$$2x_1 + 4x_2 - 10x_3 = -2$$

$$4. \quad 3x_1 + 9x_2 - 21x_3 = 0$$

$$x_1 + 5x_2 - 12x_3 = 1$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & -2 \\ 3 & 9 & -21 & 0 \\ 1 & 5 & -12 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 3 & 9 & -21 & 0 \\ 2 & 4 & -10 & -2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 0 & -6 & 15 & -3 \\ 0 & -6 & 14 & -4 \end{array} \right] \rightarrow$$

$$R_1 \leftrightarrow R_3$$

$$-3R_1 + R_2 \rightarrow R_2$$

$$-\frac{1}{6}R_2 \rightarrow R_2$$

$$-2R_1 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -12 & 1 \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & -6 & 14 & -4 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & -1 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array} \right] \rightarrow$$

$$-5R_2 + R_1 \rightarrow R_1$$

$$-R_3 \rightarrow R_3$$

$$\frac{5}{2}R_3 + R_2 \rightarrow R_2$$

$$6R_2 + R_3 \rightarrow R_3$$

$$-\frac{1}{2}R_3 + R_1 \rightarrow R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

So the only solution of the systems is $x_1 = -2, x_2 = 3, x_3 = 1$.

$$\begin{aligned} 5. \quad & 2x_1 + 4x_2 - 2x_3 = 2 \\ & -3x_1 - 6x_2 + 3x_3 = -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ -3 & -6 & 3 & -3 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -3 & -6 & 3 & -3 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\frac{1}{2}R_1 \rightarrow R_1 \qquad 3R_1 + R_2 \rightarrow R_2$$

We can't go any further, there are no contradictions, and we can't uniquely solve for the variables. This means that there are infinitely many solutions, and this time we'll introduce two parameters. Let $x_3 = t$ and $x_2 = s$. If we substitute into the top row, we get $x_1 + 2s - t = 1 \Rightarrow x_1 = t - 2s + 1$, so the solutions of the system are given by $x_3 = t, x_2 = s, x_1 = t - 2s + 1$; where t and s are any real numbers.

You might think that only systems with one solution or no solution are interesting or have practical applications, but systems with infinitely many solutions come up quite a bit in applications. Even though a system has infinitely many mathematical solutions doesn't mean that it has infinitely many practical solutions.

Example 1: A company wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15, and 20 passengers, respectively, and the leasing costs are \$8,000, \$14,000, and \$16,000, respectively. What's the cheapest way for the company to accomplish its goal?

First, we'll figure out all the different combinations of three types of airplanes the company can lease by solving a linear system of equations:

Let x_1 = the number of 10 passenger planes, x_2 = the number of 15 passenger planes, and x_3 = the number of 20 passenger planes.

$$\begin{aligned}x_1 + x_2 + x_3 &= 12 \\ 10x_1 + 15x_2 + 20x_3 &= 220\end{aligned}$$

Let's solve this system using Gauss-Jordan Elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 10 & 15 & 20 & 220 \end{array} \right] \xrightarrow{-10R_1 + R_2 \rightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 5 & 10 & 100 \end{array} \right] \xrightarrow{\frac{1}{5}R_2 \rightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 20 \end{array} \right] \xrightarrow{-R_2 + R_1 \rightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -8 \\ 0 & 1 & 2 & 20 \end{array} \right]$$

So the system has infinitely many solutions given by $x_1 = t - 8, x_2 = 20 - 2t, x_3 = t$; where t is any real number. But, x_1, x_2 , and x_3 are number of airplanes, so they have to be nonnegative whole numbers.

$$t - 8 \geq 0$$

$$8 \leq t$$

So $20 - 2t \geq 0$ and t must be a whole number. This means that $t \leq 10$, and if you

$$t \geq 0$$

$$0 \leq t$$

combine them you get $8 \leq t \leq 10$ and t is a whole number. So instead of infinitely many solutions, we actually get three of them because t must be 8, 9 or 10. So the three combinations of airplanes that they can lease are $x_1 = 0, x_2 = 4, x_3 = 8$ and

$x_1 = 1, x_2 = 2, x_3 = 9$ and $x_1 = 2, x_2 = 0, x_3 = 10$. Now we have to determine which of these three combinations is the cheapest.

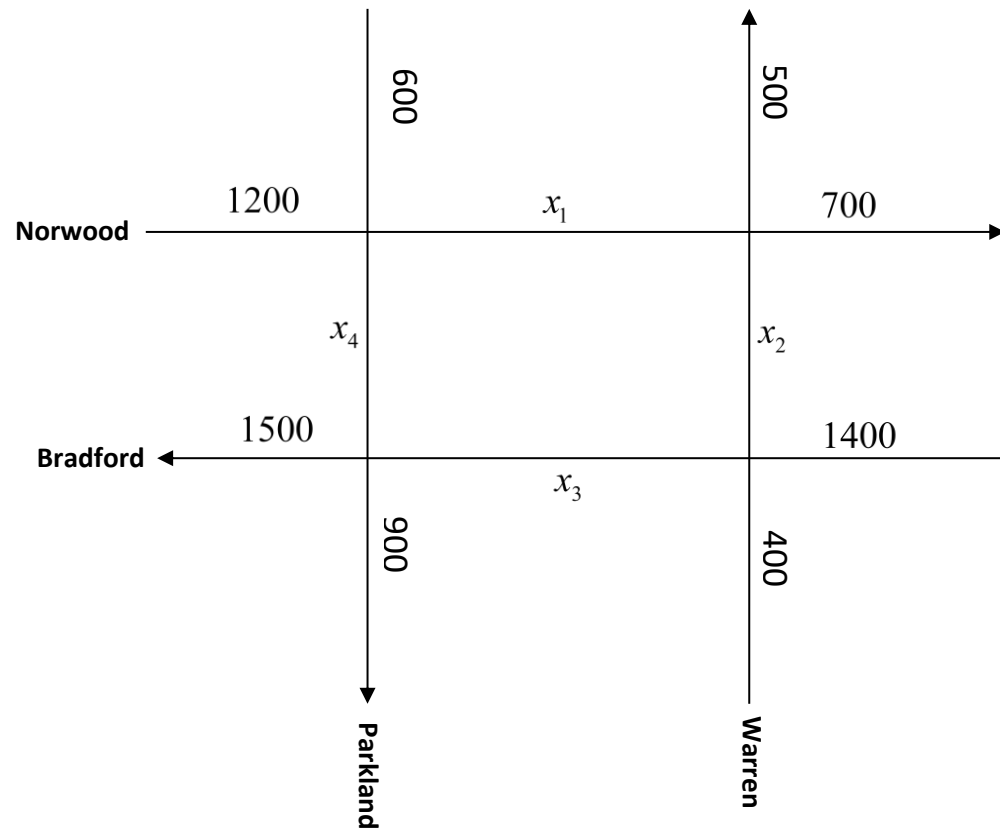
$$x_1 = 0, x_2 = 4, x_3 = 8; \text{ cost} = \$184,000$$

$$x_1 = 1, x_2 = 2, x_3 = 9; \text{ cost} = \$180,000$$

$$x_1 = 2, x_2 = 0, x_3 = 10; \text{ cost} = \$176,000$$

So the cheapest way for the company to achieve its goal is to lease 2 of the 10 passenger planes and 10 of the 20 passenger planes.

Example 2: The diagram shows the traffic flow at the intersections of four one-way streets.



The traffic rates are in cars per hour.

In order to have smooth traffic flow, the number of cars entering an intersection must equal the number of cars leaving an intersection. This leads to four equations-one for each intersection:

Intersection	Equation
Norwood and Warren	$x_1 + x_2 = 1200$
Bradford and Warren	$x_2 + x_3 = 1800$
Bradford and Parkland	$x_3 + x_4 = 2400$
Norwood and Parkland	$x_1 + x_4 = 1800$

Here's the augmented matrix corresponding to the system of linear equations.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1200 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2400 \\ 1 & 0 & 0 & 1 & 1800 \end{array} \right]$$

The result of Gauss-Jordan Elimination is the following matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1800 \\ 0 & 1 & 0 & -1 & -600 \\ 0 & 0 & 1 & 1 & 2400 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

So the mathematical solution is

$$x_4 = t, x_3 = 2400 - t, x_2 = t - 600, x_1 = 1800 - t, \text{ where } t \text{ is any real \#}$$

Since the traffic flows must be nonnegative, it must be that

$$0 \leq t$$

$$t \leq 2400$$

$$600 \leq t$$

$$t \leq 1800$$

In order for all of these inequalities to be true, it must be that $600 \leq t \leq 1800$. So the true solution of the system is

$$x_4 = t, x_3 = 2400 - t, x_2 = t - 600, x_1 = 1800 - t, \text{ where } 600 \leq t \leq 1800.$$

Here are the maximum and minimum traffic flows in the network:

Street Section	Minimum Flow	Maximum Flow
Norwood between Parkland and Warren, x_1	0	1200
Warren between Bradford and Norwood, x_2	0	1200
Bradford between Warren and Parkland, x_3	600	1800
Parkland between Bradford and Norwood, x_4	600	1800

If traffic on Warren between Bradford and Norwood is restricted to 100 cars per hour due to construction, here's the traffic flow in the rest of the system.

$$x_2 = t - 600 = 100 \Rightarrow t = 700$$

$$x_4 = 700, x_3 = 2400 - 700 = 1700, x_1 = 1800 - 700 = 1100$$

If the following tolls are charged, let's determine the least and greatest amount of money generated from the tolls per hour.

Street Section	Toll
Norwood between Parkland and Warren, x_1	\$.25
Warren between Bradford and Norwood, x_2	\$.50
Bradford between Warren and Parkland, x_3	\$.20
Parkland between Bradford and Norwood, x_4	\$.15

The total toll per hour in cents is

$$\begin{aligned} & 25x_1 + 50x_2 + 20x_3 + 15x_4 \\ &= 25(1800 - t) + 50(t - 600) + 20(2400 - t) + 15(t) \\ &= 20t + 63000; \quad 600 \leq t \leq 1800 \end{aligned}$$

So the maximum toll amount will occur for $t = 1800$, giving a maximum toll amount of 99,000 cents or \$990, and the minimum toll amount will occur for $t = 600$, giving a minimum toll amount of 75,000 cents or \$750.