Identity Matrices:

An Identity Matrix is a square matrix with 1's on its diagonal and 0's everywhere else.

Examples:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **is the** 2×2 **identity matrix.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **is the** 3×3 **identity matrix.**

Identity matrices are abbreviated as *I*, and they have the property like the number 1, that if you multiply any given matrix with *I* in either order(as long as the product is defined)you get the given matrix as the result.

Examples:

$$\underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$$

$$\underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 2 & -3 \\ 4 & 5 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 2 & -3 \\ 4 & 5 \end{bmatrix}}$$

For numbers, given a nonzero number, *a*, its multiplicative inverse(reciprocal) is the number *b* so that *ab* = *ba* = 1. For numbers, it's very easy to find the multiplicative inverse of a number.

Examples:

The multiplicative inverse of 2 is $\frac{1}{2}$. The multiplicative inverse of $\frac{2}{5}$ is $\frac{5}{2}$. And the only number that doesn't have a multiplicative inverse is 0.

For square matrices, the process of finding the multiplicative inverse of a square matrix is more complicated, and determining if a square matrix even has a multiplicative inverse is complicated as well.

Inverses of Square Matrices:

We'll work out the details for finding the inverse of a 2×2 matrix and generalize it to larger square matrices.

Suppose that $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **and we'd like to find a matrix** $B=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ **so that** $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x & y \\ z & w \end{bmatrix}}_{B}=\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$**. If we multiply out the left side, we get the matrix equation** $\underbrace{\begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}}_{AB}=\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$**. This matrix equation is equivalent to the following linear system of equations:** $\begin{aligned} ax+bz&=1 \\ ay+bw&=0 \\ cx+dz&=0 \\ cy+dw&=1 \end{aligned}$**. This system of 4 equations naturally splits into 2**

systems each of 2 equations: $\begin{matrix} ax + bz = 1 \\ cx + dz = 0 \end{matrix}$ **and** $\begin{matrix} ay + bw = 0 \\ cy + dw = 1 \end{matrix}$**. These systems can be written in augmented matrix form as** $\left[\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right]$ **in which case we're solving for *x* and *z*, and** $\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array}\right]$ **in which case we're solving for *y* and *w*. Since the row operations for solving both systems using Gauss-Jordan Elimination are the same, it's more efficient to combine them into 1 augmented matrix:** $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$**. If we can convert the left side of the vertical bar into** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**, then the right side of the vertical bar will be the inverse of *A*. If it's impossible to convert the left side of the vertical bar into** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**, then *A* doesn't have an inverse.**

So in general, given a square matrix *A*, form the augmented matrix $[A|I]$ and perform Gauss-Jordan Elimination on it. If you arrive at $[I\,|\,B]$ then *B* is the inverse of *A*. If it's impossible to get the identity matrix on the left side of the bar, then *A* doesn't have an inverse. The notation for the inverse of matrix *A* is A^{-1}.

Examples:

1. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$**.**

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \underset{-3R_1+R_2 \to R_2}{\to} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right] \underset{-R_2 \to R_2}{\to} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right] \underset{-2R_2+R_1 \to R_1}{\to} \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

2. Find the inverse of the matrix $B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \underset{-R_1+R_2\to R_2}{\rightarrow} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \underset{-2R_2+R_1\to R_1}{\rightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

So $B^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$.

3. Find the inverse of the matrix $C = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{array}\right] \underset{2R_1+R_2\to R_2}{\rightarrow} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

It’s not possible to get the identity matrix on the left side of the bar, so C doesn’t have an inverse.

4. Find the inverse of the matrix $D=\begin{bmatrix}-5 & -2 & -2\\ 2 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$**.**

$$\left[\begin{array}{ccc|ccc}-5 & -2 & -2 & 1 & 0 & 0\\ 2 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 & 1\end{array}\right]\underset{R_1\leftrightarrow R_3}{\rightarrow}\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 0 & 1\\ 2 & 1 & 0 & 0 & 1 & 0\\ -5 & -2 & -2 & 1 & 0 & 0\end{array}\right]\underset{\substack{-2R_1+R_2\rightarrow R_2\\ 5R_1+R_3\rightarrow R_3}}{\rightarrow}\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & -2 & 0 & 1 & -2\\ 0 & -2 & 3 & 1 & 0 & 5\end{array}\right]\underset{2R_2+R_3\rightarrow R_3}{\rightarrow}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & -2 & 0 & 1 & -2\\ 0 & 0 & -1 & 1 & 2 & 1\end{array}\right]\underset{-R_3\rightarrow R_3}{\rightarrow}\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & -2 & 0 & 1 & -2\\ 0 & 0 & 1 & -1 & -2 & -1\end{array}\right]\underset{\substack{2R_3+R_2\rightarrow R_2\\ -R_3+R_1\rightarrow R_1}}{\rightarrow}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 2 & 2\\ 0 & 1 & 0 & -2 & -3 & -4\\ 0 & 0 & 1 & -1 & -2 & -1\end{array}\right]$$

So $D^{-1}=\begin{bmatrix}1 & 2 & 2\\ -2 & -3 & -4\\ -1 & -2 & -1\end{bmatrix}$**.**

5. Find the inverse of the matrix $E = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$**.**

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-2R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

It's not possible to get the identity on the left side of the vertical bar, so *E* doesn't have an inverse.

If a matrix has an inverse it's said to be *invertible*. If a matrix doesn't have an inverse it's said to be *singular*.

A nice application of inverse matrices is in encoding and decoding messages.

The letters in the message are converted into numbers and placed in a matrix. This matrix of numbers is multiplied with the encoding matrix to produce a matrix of different numbers. The new numbers are sent as the encoded message. The receiver puts the encoded message into a matrix and multiplies with the decoding matrix(the inverse of the encoding matrix) to get the original numbers. The original numbers are then converted back into the letters of the message.

Example:

blank	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

The message, HELLO, would be converted into 8, 5, 12, 12, 15, and if we use a 2×2 **encoding matrix, we'll add a blank at the end to get 8, 5, 12, 12, 15, 0. Next, we'll assemble the numbers into a matrix to get** $\begin{bmatrix} 8 & 12 & 15 \\ 5 & 12 & 0 \end{bmatrix}$**. If our encoding matrix is**

$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, **then to encode the message, we'll perform the matrix product** $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 & 12 & 15 \\ 5 & 12 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 36 & 15 \\ 23 & 48 & 15 \end{bmatrix}$ **and get a coded message of 18, 23, 36, 48, 15, 15.**

To get back to the original message, we'd re-assemble the numbers into a matrix, $\begin{bmatrix} 18 & 36 & 15 \\ 23 & 48 & 15 \end{bmatrix}$ **and multiply with the decoding matrix,** $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$**.**

$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 18 & 36 & 15 \\ 23 & 48 & 15 \end{bmatrix} = \begin{bmatrix} 8 & 12 & 15 \\ 5 & 12 & 0 \end{bmatrix}$**, and this brings us back to the original message.**