Solving Matrix Equations:

For equations involving numbers, like $ax = b$, where *a* and *b* are given numbers and *x* is the variable whose value we want to determine, the solution process is to multiply both sides of the equation by the multiplicative inverse of *a*(its reciprocal).

$$a^{-1}ax = a^{-1}b \Rightarrow x = a^{-1}b$$

For matrix equations of the form $AX = B$, where *A* and *B* are given matrices and *X* is the variable matrix whose value we want to determine, the process is virtually identical. If the matrix *A* has an inverse, then you multiply both sides of the matrix equation by the inverse of *A*.

$$A^{-1}AX = A^{-1}B \Rightarrow X = A^{-1}B$$

This method of solution only works if *A* has an inverse.

Examples:

1. Solve the matrix equation $AX = B$**, where** $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$**,** $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$**.**

$$\left\{ \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \right\}$$

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix}$$

2. Solve the matrix equation $AX = B$**, where** $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$**,** $B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$**.**

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$$

Systems of Linear Equations as Matrix Equations:

A system of linear equations can be rewritten as a matrix equation. For example, the system of linear equations

$$x_1 + 2x_2 = 1$$
$$x_1 + 3x_2 = -3$$

can be written as the matrix equation $AX = B$ or $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$**, where** $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ **is called the coefficient matrix,** $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **is called the variable matrix, and** $B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ **is called the constant matrix. In the case of a square system, if the coefficient matrix has an inverse, then the matrix equation and therefore the system can be solved using the inverse matrix of the coefficient matrix. In the current example, we'd get**

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$ **and the solution of the system is** $x_1 = 9, x_2 = -4$**.**

Rewrite the following systems as matrix equations and solve using the inverse matrix method, if possible.

1. $$\begin{aligned} 2x_1 + x_2 &= 8 \\ -4x_1 + 3x_2 &= -4 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

Find the inverse of $\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$ using Gauss-Jordan Elimination or the Inverse Matrix Calculator.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -4 \end{bmatrix} = \begin{bmatrix} .3 & -.1 \\ .4 & .2 \end{bmatrix} \begin{bmatrix} 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 2.4 \end{bmatrix}$$

So $x_1 = 2.8, x_2 = 2.4$.

$$\begin{aligned} 3x_1 + 2x_3 &= 9 \\ -x_1 + 4x_2 + x_3 &= -7 \\ -2x_1 + 3x_2 &= 6 \end{aligned}$$

2.

$$\begin{bmatrix} 3 & 0 & 2 \\ -1 & 4 & 1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -7 \\ 6 \end{bmatrix}$$

Find the inverse of $\begin{bmatrix} 3 & 0 & 2 \\ -1 & 4 & 1 \\ -2 & 3 & 0 \end{bmatrix}$ using Gauss-Jordan Elimination or the Inverse Matrix Calculator.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 \\ -1 & 4 & 1 \\ -2 & 3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ -7 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 & 6 & -8 \\ -2 & 4 & -5 \\ 5 & -9 & 12 \end{bmatrix} \begin{bmatrix} 9 \\ -7 \\ 6 \end{bmatrix} = \begin{bmatrix} -117 \\ -76 \\ 180 \end{bmatrix}$$

So $x_1 = -117, x_2 = -76, x_3 = 180.$

3. $$\begin{aligned} 2x_1 + x_2 - x_3 &= 8 \\ -4x_1 + 3x_2 + x_3 &= -4 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -4 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

It can't be solved using the inverse matrix method because the coefficient matrix $\begin{bmatrix} 2 & 1 & -1 \\ -4 & 3 & 1 \end{bmatrix}$ isn't a square matrix, which means it doesn't have an inverse. The system can be solved using previous methods.

4. $$\begin{aligned} 2x_1 + x_2 &= 8 \\ -4x_1 - 2x_2 &= -4 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

It can't be solved using the inverse matrix method because the coefficient matrix $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ doesn't have an inverse. The system can be solved using previous methods.

Here's a basic application:

Parking fees at a zoo are \$5.00 for local residents and \$7.50 for all others. At the end of each day, the total number of vehicles parked that day and the gross receipts that day are recorded.

Here are the results for a recent two-day period:

	Day 1	Day 2
# of parked vehicles	1,200	1,550
Gross receipts	\$7,125	\$9,825

How many vehicles in each category used the zoo's parking on Day 1?

If $x_1 = \#\text{ of local residential parked cars}$ **and** $x_2 = \#\text{ of other parked cars}$**, then we can write the following system for Day 1:**

$$\begin{aligned} x_1 + x_2 &= 1,200 \\ 5x_1 + 7.5x_2 &= 7,125 \end{aligned}$$

And in matrix form, $\begin{bmatrix} 1 & 1 \\ 5 & 7.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1,200 \\ 7,125 \end{bmatrix}$**. So to answer this question, we can multiply the inverse of the coefficient matrix with the constant matrix.**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 7.5 \end{bmatrix}^{-1} \begin{bmatrix} 1,200 \\ 7,125 \end{bmatrix} = \begin{bmatrix} 3 & -.4 \\ -2 & .4 \end{bmatrix} \begin{bmatrix} 1,200 \\ 7,125 \end{bmatrix} = \begin{bmatrix} 750 \\ 450 \end{bmatrix}$$

So now we know that there were 750 local residents parked and 450 in the other category parked.

If you wanted to know the same information about Day 2, you could do the same process again, just with the new constant matrix:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -.4 \\ -2 & .4 \end{bmatrix} \begin{bmatrix} 1,550 \\ 9,825 \end{bmatrix} = \begin{bmatrix} 720 \\ 830 \end{bmatrix}$$

So now we know that there were 720 local residents parked and 830 in the other category parked.