Math 1324 Review 2(answers)

1. Solve the following system using Gauss-Jordan Elimination.

$$x+y+2z=6$$
$$-x-2z=-5$$
$$2x+2y+4z=12$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ -1 & 0 & -2 & -5 \\ 2 & 2 & 4 & 12 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1+R_2 \to R_2$

$-2R_1+R_3 \to R_3$

$-R_2+R_1 \to R_1$

Infinitely many solutions given by $\begin{array}{l} z=t \\ y=1 \\ x=5-2t \end{array}$ where t is any real number.

2. Solve the following system using Gauss-Jordan Elimination.

$$x-y+2z=4$$
$$2x+3y-z=1$$
$$7x+3y+4z=7$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & 3 & -1 & 1 \\ 7 & 3 & 4 & 7 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & -7 \\ 0 & 10 & -10 & -21 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -\frac{7}{5} \\ 0 & 10 & -10 & -21 \end{array}\right]$$

$-2R_1+R_2 \to R_2$

$-7R_1+R_3 \to R_3$

$\frac{1}{5}R_2 \to R_2$

$R_2+R_1 \to R_1$

$-10R_2+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{13}{5} \\ 0 & 1 & -1 & -\frac{7}{5} \\ 0 & 0 & 0 & -7 \end{array}\right]$$

The system has no solution.

3. a) Solve the following system using Gauss-Jordan Elimination.

$$x + y + z = 9$$
$$x + 5y + 10z = 44$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 5 & 10 & 44 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 4 & 9 & 35 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & \frac{9}{4} & \frac{35}{4} \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & \frac{1}{4} \\ 0 & 1 & \frac{9}{4} & \frac{35}{4} \end{array}\right]$$

$-R_1 + R_2 \to R_2$ $\qquad \frac{1}{4}R_2 \to R_2$ $\qquad -R_2 + R_1 \to R_1$

The system has infinitely many solutions given by $z = t$, $y = \dfrac{35 - 9t}{4}$, $x = \dfrac{1 + 5t}{4}$ where t is any real number.

b) Find positive integers x, y, and z that solve the system.

In order for the system to have positive integer solutions t must be a positive integer and $35 - 9t$ and $1 + 5t$ must be positive multiples of 4.

t	$35-9t$	$1+5t$
1	26	6
2	17	11
3	8	16

So t must equal 3 and we get the solution $\boxed{x = 4, y = 2, z = 3}$

4. Find the value(s) of C, if possible, so that the system of equations with the given augmented matrix has

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & C & 9 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & C & 9 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 2 \\ 4 & C & 9 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 2 \\ 0 & C-6 & 1 \end{array}\right]$$

$\frac{1}{2}R_1 \to R_1$ $\qquad -4R_1 + R_2 \to R_2$

a) exactly one solution
$C \neq 6$

b) infinitely many solutions
Not possible

c) no solution
$C = 6$

5. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & B & 0 \end{array}\right]$$

a) Is it possible for the system to have no solution? Explain.

No. Since the right sides of the equations are all zeros, $x=0, y=0, z=0$ will always be a solution.

b) For what value(s) of B will the system have infinitely many solutions?

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & B & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & B+1 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & B-2 & 0 \end{array}\right]$$

$-2R_1 + R_2 \to R_2$

$R_1 + R_3 \to R_3$

$-3R_2 + R_3 \to R_3$

The system will have infinitely many solutions if $B = 2$.

6. Solve $AX = B$ for X if $A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

$AX = B$, so $A^{-1}AX = A^{-1}B$, or $X = A^{-1}B$

$$A^{-1}B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 19 \\ 23 \end{bmatrix}}$$

7. Given that $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 8 & -2 \\ -6 & 4 \end{bmatrix}$, find the matrix X that satisfies the equation $2X + B = -3A + C$.

$2X + B = -3A + C$, so

$$X = \tfrac{1}{2}(-3A + C - B) = \tfrac{1}{2}\left(\begin{bmatrix} -3 & 6 \\ 9 & -12 \end{bmatrix} + \begin{bmatrix} 8 & -2 \\ -6 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix}\right)$$

$$= \tfrac{1}{2}\begin{bmatrix} 5 & 3 \\ -2 & -6 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ -1 & -3 \end{bmatrix}}$$

8. a) Find the inverse of the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ by performing Gauss-Jordan on the augmented matrix $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$-2R_1 + R_2 \to R_2 \qquad -R_2 + R_1 \to R_1$

$$\text{so } \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}}$$

b) Solve the system $\begin{aligned} x + y &= 2 \\ 2x + 3y &= 8 \end{aligned}$ using the inverse matrix method.

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \text{ so } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\boxed{x = -2, y = 4}$$

c) Solve the system $\begin{aligned} x + y &= 3 \\ 2x + 3y &= -2 \end{aligned}$ using the inverse matrix method.

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \text{ so } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ -8 \end{bmatrix}$$

$$\boxed{x = 11, y = -8}$$

d) Solve the system $\begin{aligned} x + y &= 1 \\ 2x + 3y &= 1 \end{aligned}$ using the inverse matrix method.

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ so } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\boxed{x = 2, y = -1}$$

9. The following matrix gives the number of direct flights among the four cities A, B, C, and D.

Destination

Origin	A	B	C	D
A	0	1	0	1
B	1	0	0	1
C	0	0	0	1
D	1	1	1	0

a) Find the number of one-stop flights from city A to city C.

This would be the $(1,3)$ entry of the square of the matrix

$$\begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}^2=\begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}\begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}$$

$$=\begin{bmatrix}2&1&\boxed{1}&1\\1&2&1&1\\1&1&1&0\\1&1&0&3\end{bmatrix}$$

So there is 1 one-stop flight from city A to city C.

b) Find the total number of flights from city B to city C that are either direct or one-stop.

This would be the $(2,3)$ entry of the sum of the matrix and its square.

$$\begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}+\begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}^2=\begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}+\begin{bmatrix}2&1&1&1\\1&2&1&1\\1&1&1&0\\1&1&0&3\end{bmatrix}$$

$$=\begin{bmatrix}2&2&1&2\\2&2&\boxed{1}&2\\1&1&1&1\\2&2&1&3\end{bmatrix}$$

So there is 1 flight from city B to city C which is either direct or one-stop.

c) Find the matrix that gives the number of two-stop flights among these cities.

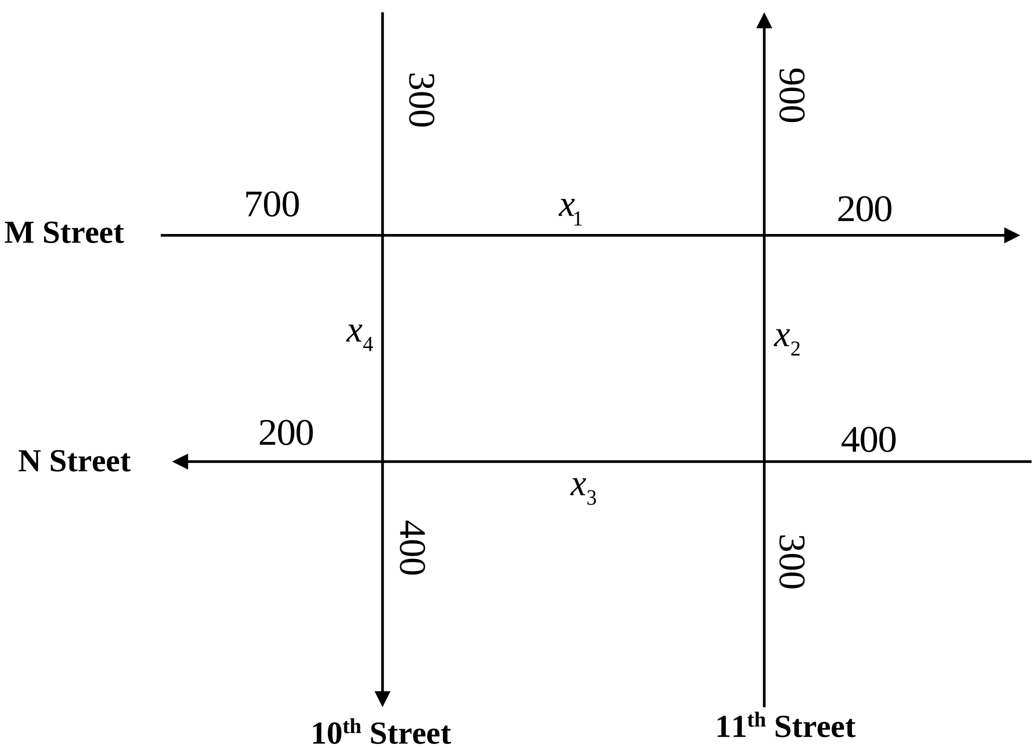
This would be the matrix raised to the third power.

$$\begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}^3 = \begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}\begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}^2$$

$$= \begin{bmatrix}0&1&0&1\\1&0&0&1\\0&0&0&1\\1&1&1&0\end{bmatrix}\begin{bmatrix}2&1&1&1\\1&2&1&1\\1&1&1&0\\1&1&0&3\end{bmatrix}$$

$$= \boxed{\begin{bmatrix}2&3&1&4\\3&2&1&4\\1&1&0&3\\4&4&3&2\end{bmatrix}}$$

10. The diagram shows the traffic flow at the intersections of four one-way streets.



The traffic rates are in cars per hour.

In order to have smooth traffic flow, the number of cars entering an intersection must equal the number of cars leaving an intersection. This leads to four equations, one for each intersection:

a) Complete the table of intersection equations.

Intersection	Equation
M Street and 11th Street	$x_1 + x_2 = 1100$
N Street and 11th Street	$x_2 + x_3 = 700$
N Street and 10th Street	$x_3 + x_4 = 600$
M Street and 10th Street	$x_1 + x_4 = 1000$

b) The augmented matrix for solving the system of equations using Gauss-Jordan Elimination is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1100 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 600 \\ 1 & 0 & 0 & 1 & 1000 \end{array}\right].$$

Complete Gauss-Jordan Elimination on the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 1 & 1 & 600 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 1 & 1 & 600 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$-R_3 + R_4 \to R_4$

So the system has infinitely many solutions given by $\begin{array}{l} x_4 = t \\ x_3 = 600 - t \\ x_2 = 100 + t \\ x_1 = 1000 - t \end{array}$ where t is any real number.

c) Since the values of $x_1, x_2, x_3,$ and x_4 must be nonnegative, write the four inequalities associated with legitimate solutions, and express the solutions in terms of a parameter with an inequality.

$$\begin{array}{l} t \geq 0 \\ 600 - t \geq 0 \\ 100 + t \geq 0 \\ 1000 - t \geq 0 \end{array} \Rightarrow \begin{array}{r} 0 \leq t \\ t \leq 600 \\ -100 \leq t \\ t \leq 1000 \end{array} \Rightarrow 0 \leq t \leq 600$$

So the solutions of the traffic problem are given by $\begin{array}{l} x_4 = t \\ x_3 = 600 - t \\ x_2 = 100 + t \\ x_1 = 1000 - t \end{array}$ where t satisfies $0 \leq t \leq 600$.

d) Determine the maximum and minimum traffic flows on the following street sections:

Street Section	Minimum Flow	Maximum Flow
M Street between 10th and 11th	400	1000
11th between M Street and N Street	100	700
N Street between 10th and 11th	0	600
10th between M Street and N Street	0	600

e) If traffic on 11th between M Street and N Street is restricted to 300 cars per hour due to construction, determine the traffic flow in the rest of the system.

$x_2 = 300$, which means that $t = 200$, so $\boxed{x_1 = 800, x_3 = 400, x_4 = 200}$

f) If the following tolls are charged, determine the least and greatest amount of money generated from the tolls.

$$\begin{aligned} Money &= 25x_1 + 50x_2 + 20x_3 + 15x_4 \\ &= 25(1000 - t) + 50(100 + t) + 20(600 - t) + 15t \\ &= 42000 + 20t \end{aligned}$$

Street Section	Toll	Minimum	Maximum
M Street between 10th and 11th	**$.25**	$420	$540
11th between M Street and N Street	**$.50**		
N Street between 10th and 11th	**$.20**		
10th between M Street and N Street	**$.15**		

11. A State Fish and Game Department will supply three types of food to a lake that can support three species of fish. Each fish of Species 1 consumes, each week, an average of 1 unit of Food 1, 1 unit of Food 2, and 2 units of Food 3. Each fish of Species 2 consumes, each week, an average of 3 units of Food 1, 4 units of Food 2, and 5 units of Food 3. For a fish of Species 3, the average weekly consumption is 2 units of Food 1, 1 unit of Food 2, and 5 units of Food 3. Each week 25,000 units of Food 1, 20,000 units of Food 2, and 55,000 units of Food 3 are supplied to the lake. If we asume that all food is eaten, we'd like to know how many of each type of fish can coexist in the lake.

Food 1 Equation	$x_1 + 3x_2 + 2x_3 = 25{,}000$
Food 2 Equation	$x_1 + 4x_2 + x_3 = 20{,}000$
Food 3 Equation	$2x_1 + 5x_2 + 5x_3 = 55{,}000$

a) Complete the augmented matrix for the system of equations.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 25{,}000 \\ 1 & 4 & 1 & 20{,}000 \\ 2 & 5 & 5 & 55{,}000 \end{array}\right]$$

b) The result of performing Gauss-Jordan Elimination on the augmented matrix is the following:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 40,000 \\ 0 & 1 & -1 & -5,000 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the values of x_1, x_2, and x_3 must be nonnegative whole numbers, write the four inequalities associated with legitimate solutions, and express the solutions in terms of a parameter with an inequality.

$$\begin{array}{lllll} t \geq 0 & & 0 \leq t & & \\ t - 5,000 \geq 0 & \Rightarrow & 5,000 \leq t & \Rightarrow & 5,000 \leq t \leq 8,000 \\ 40,000 - 5t \geq 0 & & t \leq 8,000 & & \end{array}$$

So the solutions are given by $\begin{array}{l} x_3 = t \\ x_2 = t - 5,000 \\ x_1 = 40,000 - 5t \end{array}$ where t is a whole number that satisfies $5,000 \leq t \leq 8,000$.

c) What is the largest number of fish Species 2 that can survive in the lake?

3,000

d) What is the smallest number of fish Species 3 that can survive in the lake?

5,000

e) What is the largest total population of all three species that can coexist in the lake?

$$\begin{aligned} Total &= t + t - 5,000 + 40,000 - 5t \\ &= 35,000 - 3t \end{aligned}$$

So the largest total population is 20,000.

f) What is the smallest total population of all three species that can coexist in the lake?

$$\begin{aligned} Total &= t + t - 5,000 + 40,000 - 5t \\ &= 35,000 - 3t \end{aligned}$$

So the smallest total population is 11,000.