Independence and Dependence:

The events *E* and *F* are independent if any of the following three conditions are true.

$$P(E \mid F) = P(E)$$

$$P(F \mid E) = P(F)$$

$$P(E \cap F) = P(E) \cdot P(F)$$

Events that are not independent are dependent.

$P(E) = .5$**,** $P(F) = .6$**, and** $P(E \cap F) = .3$

$P(E) \cdot P(F) = (.5)(.6) = .3 = P(E \cap F)$, so E and F are independent.

$P(A) = .7$**,** $P(B) = .6$**, and** $P(A \cap B) = .4$

$P(A) \cdot P(B) = (.7)(.6) = .42 \neq P(A \cap B)$, so A and B are dependent.

Certain events are always assumed to be independent, such as, flips of a coin and rolls of a die.

Odds and Probability:

The odds in favor of an event *E* is the ratio of the probability that *E* will occur to the probability that *E* won't occur.

Odds in favor of E: $P(E):P(E')$ **or** $P(E)$ to $P(E')$

The odds are usually expressed as a ratio of whole numbers.

Example:

If $P(E)=\frac{2}{5}$**, then find the odds in favor of *E*.**

$$\frac{2}{5} \text{ to } \frac{3}{5}\text{, multiply by 5 to get } \boxed{2 \text{ to } 3}$$

The odds against an event E is the ratio of the probability that *E* won't occur to the probability that *E* will occur, i.e. the reversal of the odds in favor.

Odds against E: $P(E'):P(E)$ **or** $P(E')$ to $P(E)$

Example:

If $P(E)=\frac{3}{7}$**, then find the odds against *E*.**

$$\frac{4}{7} \text{ to } \frac{3}{7}\text{, multiply by 7 to get } \boxed{4 \text{ to } 3}$$

Sometimes you'll want to go from odds to probability. If the odds in favor of *E* is *a* to *b*, then $\frac{P(E)}{P(E')}=\frac{a}{b}\Rightarrow\frac{P(E)}{1-P(E)}=\frac{a}{b}$**. Cross-multiplying leads to**

$$bP(E)=a-aP(E)\Rightarrow(a+b)P(E)=a\Rightarrow P(E)=\frac{a}{a+b}.$$

Example:

If the odds in favor of *E* is 4 to 7, then find $P(E)$**.**

$$P(E)=\frac{4}{4+7}=\boxed{\frac{4}{11}}$$

Random Variable:

A random variable is a variable whose value is determined by the outcome of an experiment.

Example:

Flip a fair coin twice, and let *X* be the number of tails that occur.

$$S = \{HH, HT, TH, TT\}$$

X	**0**	1	2

Probability Distribution of a Random Variable:

It's a table of all the values of a random variable along with their associated probabilities.

Example: Using the previous random variable,

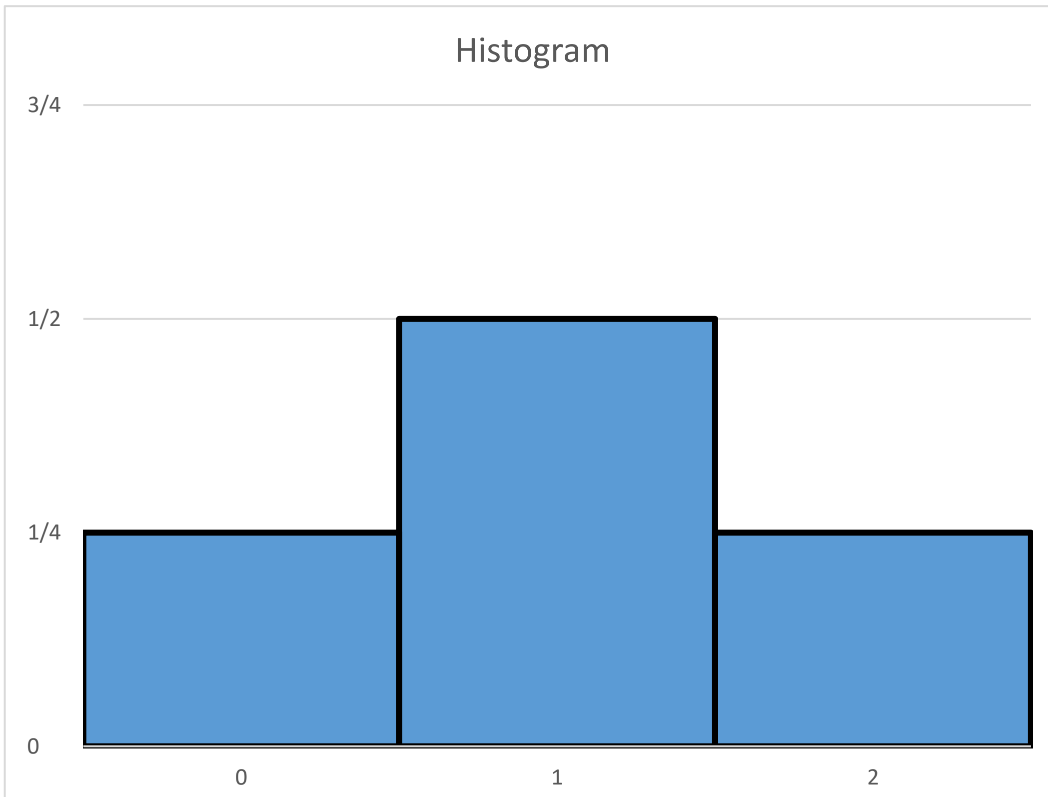
$$S = \{HH, HT, TH, TT\}$$

X	**0**	**1**	**2**
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Histogram of a Random Variable:

It’s a special bar graph of the probability distribution where the bars are centered at the values of the random variable, adjacent bars touch, and the area of the bar is equal to the probability that the random variable takes on that particular value.

Example: Using the previous random variable and distribution,



Expected Value of a Random Variable:

For *X* a random variable with probability distribution

X	x_1	x_2	x_3	$\cdots$	x_n
P(*X*)	p_1	p_2	p_3	$\cdots$	p_n

, the expected value of *X*, E(*X*), is defined as $E(X) = x_1 p_1 + x_2 p_2 + x_3 p_3 + \cdots + x_n p_n$**.**

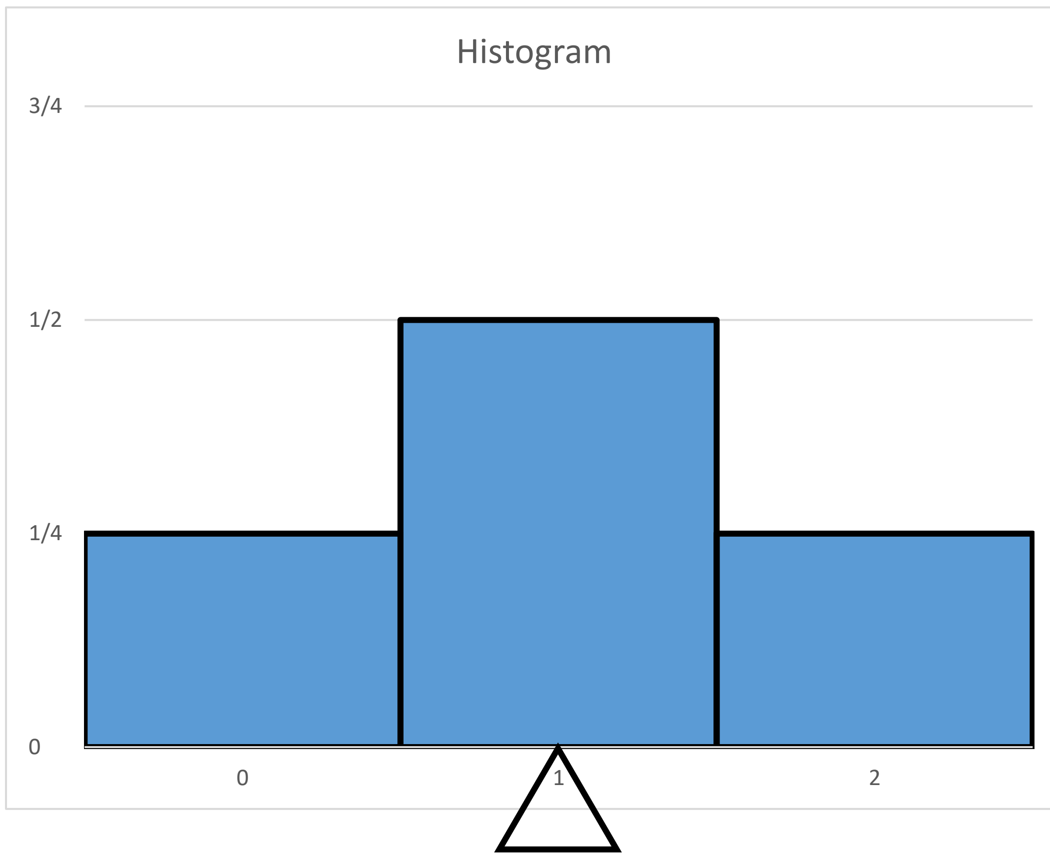
Example: Using the previous random variable and distribution,

X	**0**	**1**	**2**
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$E(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 0 + \frac{1}{2} + \frac{1}{2} = \boxed{1}$$

What does E(X) mean or represent?

Geometrically, it's the balance position of the histogram:



More practically, it represents the long-term average value of *X*.

If the experiment is performed *n* times with *X* taking on the values $X_1, X_2, \ldots, X_n$**,**

then the average value of *X* would be $\dfrac{X_1 + X_2 + \ldots + X_n}{n}$**.**

For large values of *n*, the average value of *X* will be close to E(*X*), and the bigger *n* gets, the closer the average value of *X* will get to E(*X*).

So in our example, if the experiment of flipping a fair coin twice and recording the number of tails is done a bunch of times, the average number of tails per trial will be close to 1.

Examples:

1. In a game, you roll a fair die. If you roll a 1, you win \$1; if you roll a 2, you win \$2; if you roll a 3, you win \$3. If you roll a 4, 5, or 6, you win nothing. Determine your expected winnings from one play of the game.

Let *X* be your winnings from one play of the game.

X	**\$1**	**\$2**	**\$3**	**\$0**
P(*X*)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$

$$E(X) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 0\cdot\frac{1}{2} = \boxed{\$1}$$

Interpret the expected value.

If you play this game many times, your average winnings per play will be close to \$1.

2. In a raffle, there is 1 prize of $1,000, 2 prizes of $500, 4 prizes of $250, and 10 prizes of $100. One-thousand tickets are sold at $10 each. The prizes are awarded by randomly selecting the tickets. Determine your expected net winnings from one ticket.

Let *X* be your net winnings from one ticket.

X	**$990**	**$490**	**$240**	**$90**	**-$10**
P(*X*)	$\frac{1}{1,000}$	$\frac{2}{1,000}$	$\frac{4}{1,000}$	$\frac{10}{1,000}$	$\frac{983}{1,000}$

$$E(X) = \frac{990}{1,000} + \frac{980}{1,000} + \frac{960}{1,000} + \frac{900}{1,000} - \frac{9,830}{1,000} = -\frac{6,000}{1,000} = \boxed{-\$6.00}$$

Interpret the expected value.

If you play this raffle many times, then on average, you will lose about $6.00 per play.

Another option:

Let *Y* be your gross winnings from one ticket.

Y	**$1,000**	**$500**	**$250**	**$100**	**$0**
P(*Y*)	$\frac{1}{1,000}$	$\frac{2}{1,000}$	$\frac{4}{1,000}$	$\frac{10}{1,000}$	$\frac{983}{1,000}$

$$E\left(X\right) = E\left(Y\right) - \$10$$

$$= 1 + 1 + 1 + 1 + 0 - 10 = \boxed{-\$6.00}$$

3. A 25-year-old can purchase a $10,000 one-year life insurance policy for $100. The probability of a person dying at age 25 is .002. Determine the expected profit to the insurance company from the sale of one policy.

Let *X* be the profit from the sale of one policy for the insurance company.

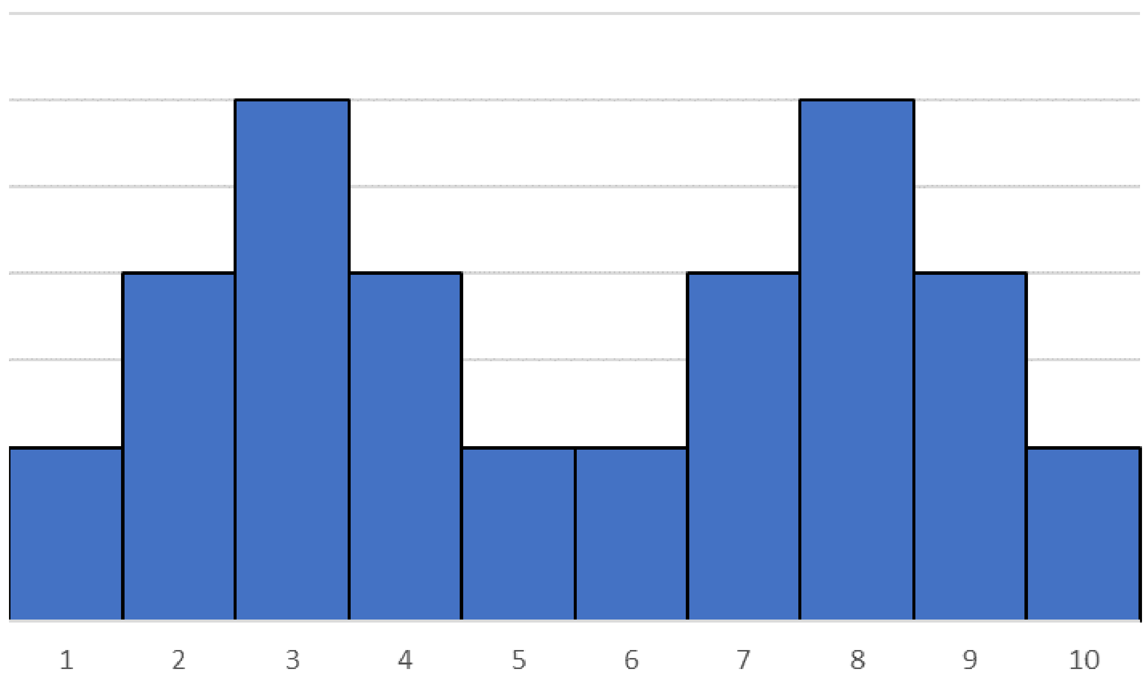
X	**-$9,900**	**$100**
P(*X*)	**.002**	.998

$$E(X) = -19.8 + 99.8 = \boxed{\$80}$$

Interpret the expected value.

If the insurance company sells many of these policies, then on average, their profit per policy will be about $80.

4. Determine the expected value of the random variable with the following histogram.



The horizontal balance position, and therefore the expected value, is 5.5.