Counting Methods:

Example:

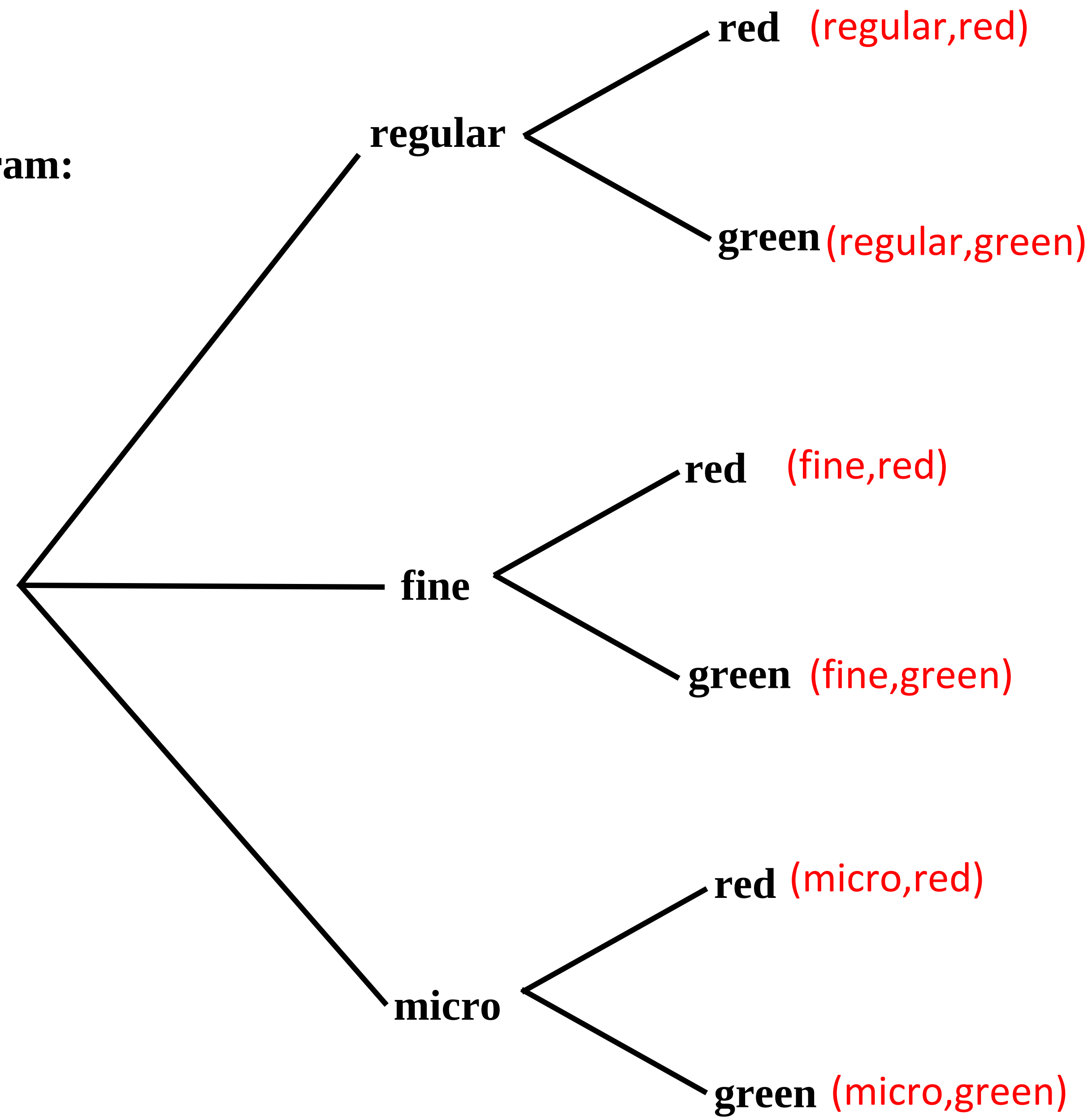
A pen has tip options of regular tip, fine tip, or micro tip, and it has ink color options of red ink or green ink. How many different pens are possible?

Using a table:

	regular	**fine**	**micro**
red	red, regular	red, fine	red, micro
green	green, regular	green, fine	green, micro

The number of pens possible is the number of cells in the table: $3 \times 2 = 6$**.**

Using a Tree Diagram:



The number of pens possible is the number of branch tips on the right: $3 \times 2 = 6$.

The Fundamental Counting Principle/Property:

If a sequence of decisions is to be made, then the number of different ways of making all the decisions is the product of the number of options for each decision.

Examples:

1. A meal consists of 1 of 8 appetizers, 1 of 10 entrees, and 1 of 5 desserts. How many different meals are possible?

8	10	5
Which appetizer?	Which entrée?	Which dessert?

The product, $8 \times 10 \times 5$, is 400.

2. In a race with 5 horses, how many different first, second, and third place finishes are possible?

5	4	3
1st place	2nd place	3rd place

The product, $5 \times 4 \times 3$, is 60.

3. In a certain small state, license plates consist of three letters followed by two digits.

a) How many different plates are possible?

26	26	26	10	10	10
letter	letter	letter	digit	digit	digit

The product, $26\times26\times26\times10\times10\times10$, is 17,576,000.

b) How many if letters can't repeat?

26	25	24	10	10	10
letter	letter	letter	digit	digit	digit

The product, $26\times25\times24\times10\times10\times10$, is 15,600,000.

c) How many if digits can't repeat?

26	26	26	10	9	8
letter	letter	letter	digit	digit	digit

The product, $26\times26\times26\times10\times9\times8$, is 12,654,720.

d) How many if no repeats?

26	25	24	10	9	8
letter	letter	letter	digit	digit	digit

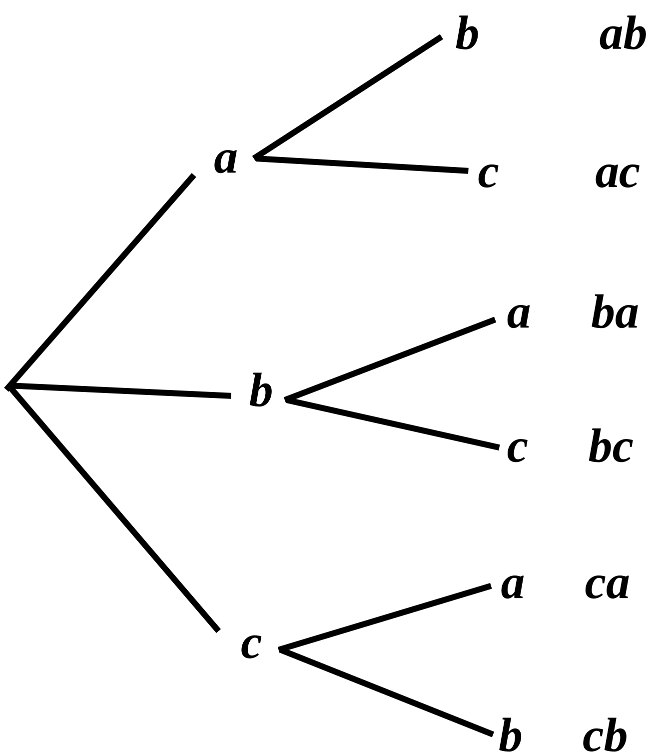
The product, $26\times25\times24\times10\times9\times8$, is 11,232,000.

<u>Permutations:</u>

A permutation is an arrangement of objects in a particular order.

Example:

Find all the permutations of the objects $\{a,b,c\}$ of size 2.



There are 6 permutations of size 2 from the 3 objects. If we just wanted to know how many, we could have used the Fundamental Counting Principle.

3	2
1^{st} position	2^{nd} position

The product, 3×2, is 6.

In general, the number of permutations of size *r* from *n* objects is abbreviated as ${}_nP_r$**. So far, we know that** ${}_3P_2 = 6$**. There's a nice formula for the value of** ${}_nP_r$ **in general, but it involves things called factorials.**

Factorials:

$$n! = n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot 2\cdot 1 \textbf{ or } n! = 1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)\cdot n$$

So $1! = 1$**.**

$$2! = 2\cdot 1 = 2$$

$$3! = 3\cdot 2\cdot 1 = 6$$

$$4! = 4\cdot 3\cdot 2\cdot 1 = 4\cdot 3! = 4\cdot 6 = 24$$

$$5! = 5\cdot 4! = 5\cdot 24 = 120$$

$$6! = 6\cdot 5! = 6\cdot 120 = 720$$

By special definition, $0! = 1$**.**

$$_{n}P_{r} = \frac{n!}{(n-r)!}$$

Let's check it out for $_{3}P_{2}$, which we already know is equal to 6.

$$_{3}P_{2} = \frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{6}{1} = 6$$

Examples:

1. Five solo singers are to perform their acts at a nightclub on Saturday night. How many different orders of their appearances are possible?

$$_{5}P_{5} = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{120}{1} = \boxed{120}$$

Or

Fundamental Counting Principle

5	4	3	2	1
1st act	2nd act	3rd act	4th act	5th act

The product, $5 \times 4 \times 3 \times 2 \times 1$, is 120.

2. From a group of 6 people, a president, vice-president, and secretary will be selected, how many different selections are possible?

$$_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\cdot 5\cdot 4\cdot 3!}{3!} = 6\cdot 5\cdot 4 = \boxed{120}$$

Or

Fundamental Counting Principle

6	5	4
president	vice-president	secretary

The product, $6\times 5\times 4$, is 120.

3. In a race with 8 horses, how many different first, second, and third place finishes are possible?

$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8\cdot 7\cdot 6\cdot 5!}{5!} = 8\cdot 7\cdot 6 = \boxed{336}$$

Or

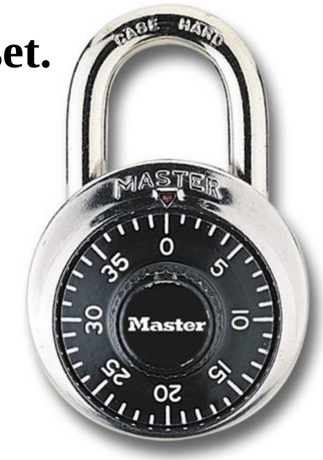
Fundamental Counting Principle

8	7	6
1^{st} place	2^{nd} place	3^{rd} place

The product, $8\times 7\times 6$, is 336.

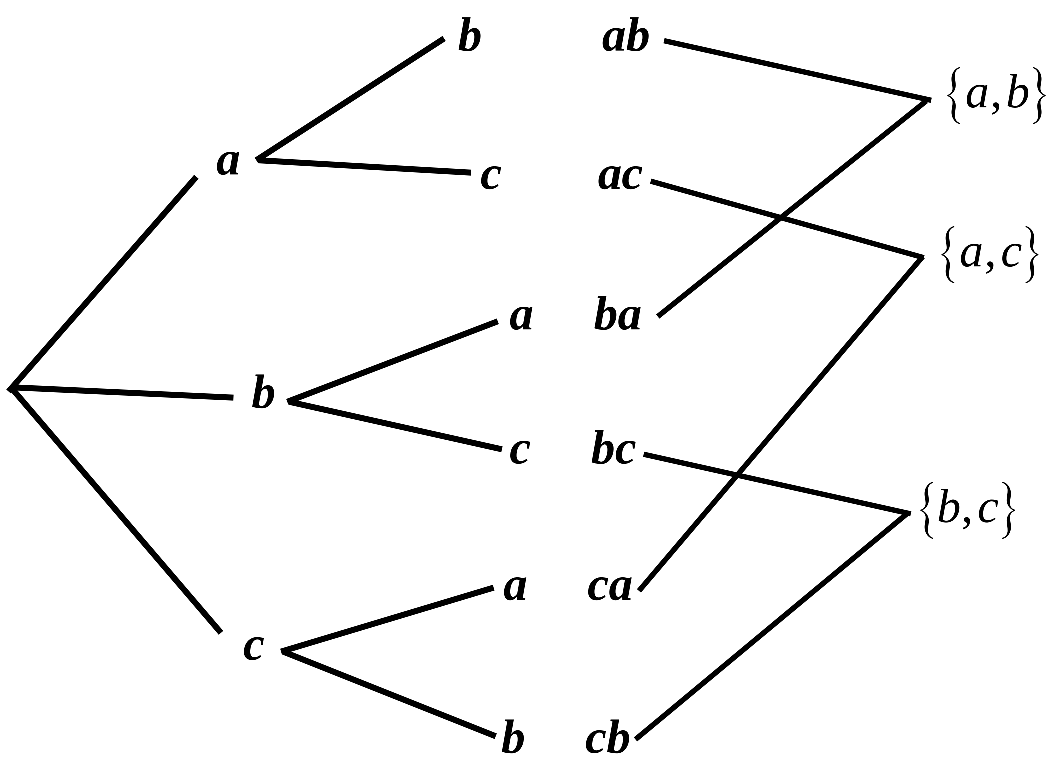
Combinations:

A combination is a selection of objects without regard to order, i.e. a subset.



Example:

Find all the combinations of the objects $\{a,b,c\}$ **of size 2.**



There are 6 permutations of size 2 from the 3 objects, but only 3 combinations of size 2 from the 3 objects. Generally, the number of combinations will be smaller than the number of permutations.

In general, the number of combinations of size *r* from *n* objects is abbreviated as $_nC_r$. So far, we know that $_3C_2 = 3$, and $_3C_2 = 3 = \frac{6}{2} = \frac{_3P_2}{2!}$. This is true in general, and leads to a nice formula for $_nC_r$. $_nC_r = \frac{n!}{r!\cdot(n-r)!}$

<u>Examples:</u>

1. A three-person committee is to be selected from a group of 10 people. How many different committees are possible?

$$_{10}C_3 = \frac{10!}{3!\cdot 7!} = \frac{10\cdot 9\cdot 8\cdot 7!}{3!\cdot 7!} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = \boxed{120}$$

2. In a certain lottery, you must select 6 numbers from the numbers 1-50. How many different lottery selections are possible?

$$_{50}C_6 = \frac{50!}{6!\cdot 44!} = \boxed{15{,}890{,}700}$$

3. **A group consists of 7 men and 8 women. A committee of 4 people will be selected.**

a) How many different 4-person committees are possible?

$$ {}_{15}C_4 = \frac{15!}{4!\cdot 11!} = \frac{15\cdot 14\cdot 13\cdot 12\cdot 11!}{4!\cdot 11!} = \frac{15\cdot 14\cdot 13\cdot 12}{4\cdot 3\cdot 2\cdot 1} = \boxed{1{,}365} $$

b) How many different 4-person committees consisting of 4 women are possible?

$$ {}_{8}C_4 = \frac{8!}{4!\cdot 4!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!\cdot 4!} = \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1} = \boxed{70} $$

c) How many different 4-person committees consisting of 3 women and 1 man are possible?

${}_{8}C_3 = 56$	${}_{7}C_1 = 7$
Which 3 women?	Which 1 man?

The product, 56×7, is 392.

d) How many different 4-person committees consisting of 2 women and 2 men are possible?

${}_{8}C_2 = 28$	${}_{7}C_2 = 21$
Which 2 women?	Which 2 men?

The product, 28×21, is 588.

e) How many different 4-person committees have at least 1 man?

If we subtract the number of committees with no men from the total number of committees, we'll have the answer. 1,365 – 70 = 1295.

Finding Probabilities Using Counting Techniques:

In the case of the equally likely assumption, $P(E) = \dfrac{n(E)}{n(S)}$**.**

Examples:

1. Allen, Bob, Carl, and David will be seated in a row of 4 chairs at random.

a) How many different ways can they be seated?

4	3	2	1
1^{st} chair	2^{nd} chair	3^{rd} chair	4^{th} chair

The product, $4 \times 3 \times 2 \times 1$, is 24.

b) How many different ways can Carl be seated in the first chair and Allen in the fourth chair?

1	2	1	1
1^{st} chair	2^{nd} chair	3^{rd} chair	4^{th} chair

The product, $1 \times 2 \times 1 \times 1$, is 2.

c) What's the probability that Carl will be seated in the first chair and Allen in the fourth chair?

$$\frac{2}{24} = \frac{1}{12}$$

2. A group consists of 4 men and 5 women. Three people will be selected at random to attend a conference.

a) How many different selections of 3 people from the group are possible?

$$ {}_9C_3 = \boxed{84} $$

b) What's the probability that the 3 people selected are all women?

$$ \frac{{}_5C_3}{{}_9C_3} = \frac{10}{84} = \boxed{\frac{5}{42}} $$

c) What's the probability that the 3 people selected are all men?

$$ \frac{{}_4C_3}{{}_9C_3} = \frac{4}{84} = \boxed{\frac{1}{21}} $$

d) What's the probability that the 3 people selected will consist of 2 women and 1 man?

$$ \frac{{}_5C_2 \cdot {}_4C_1}{{}_9C_3} = \frac{10 \cdot 4}{84} = \frac{40}{84} = \boxed{\frac{10}{21}} $$

3. Three cards will be randomly selected from a 52-card deck without replacement.

a) What's the probability that it will consist of all hearts?

$$\frac{{}_{13}C_3}{{}_{52}C_3} = \frac{286}{22{,}100} = \boxed{\frac{11}{850}}$$

b) What's the probability that it will consist of exactly 2 aces?

How many ways can you select exactly 2 aces?

${}_4C_2 = 6$	${}_{48}C_1 = 48$
Which 2 aces?	Which 1 non-ace?

$$\frac{{}_4C_2 \cdot {}_{48}C_1}{{}_{52}C_3} = \frac{6 \cdot 48}{22{,}100} = \frac{288}{22{,}100} = \boxed{\frac{72}{5{,}525}}$$

c) What's the probability that it will consist of 2 aces and a king?

$$\frac{{}_4C_2 \cdot {}_4C_1}{{}_{52}C_3} = \frac{6 \cdot 4}{22{,}100} = \frac{24}{22{,}100} = \boxed{\frac{6}{5{,}525}}$$