Conditional Probability:

Sometimes additional information is known or assumed about the outcome of an experiment.

This additional information may have an effect on the probabilities of events.

The new probability of an event *E* assuming that event *F* will occur is denoted by $P(E \mid F)$ and is called the conditional probability that *E* will occur given that *F* will occur.

Conditional probabilities are calculated using either the Reduced Sample Space Method or the Conditional Probability Formula.

Examples:

1. A fair coin is flipped twice.

$$S=\{HH,HT,TH,TT\}$$

$E=\{HH,HT,TH\}$, $F=\{HT,TH,TT\}$, $G=\{HH,TT\}$, **and** $J=\{HT,TH\}$.

Find $P(E)=\frac{3}{4},P(F)=\frac{3}{4},P(G)=\frac{1}{2},P(J)=\frac{1}{2}$.

Find $P(E \mid F)$ using Reduced Sample Space and Conditional Probability Formula.

In an equally likely sample space, $P(A \mid B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{\text{\# of outcomes in } A \cap B}{\text{\# of outcomes in } B}$**.**

$$P(E \mid F) = \frac{2}{3}$$

The original sample space, *S*, has been reduced to the new sample space, *F*.

In general, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$**.**

$$P(E \mid F) = \frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3}$$

Here we've just formed a ratio of ordinary probabilities.

Find the following conditional probabilities using the Reduced Sample Space Method.

$P(F \mid E)$

$$\frac{2}{3}$$

$P(E \mid G)$

$$\frac{1}{2}$$

$P(G \mid E)$

$$\frac{1}{3}$$

$P(G \mid J)$

$$\frac{0}{2} = 0$$

$P(F \mid J)$

$$\frac{2}{2} = 1$$

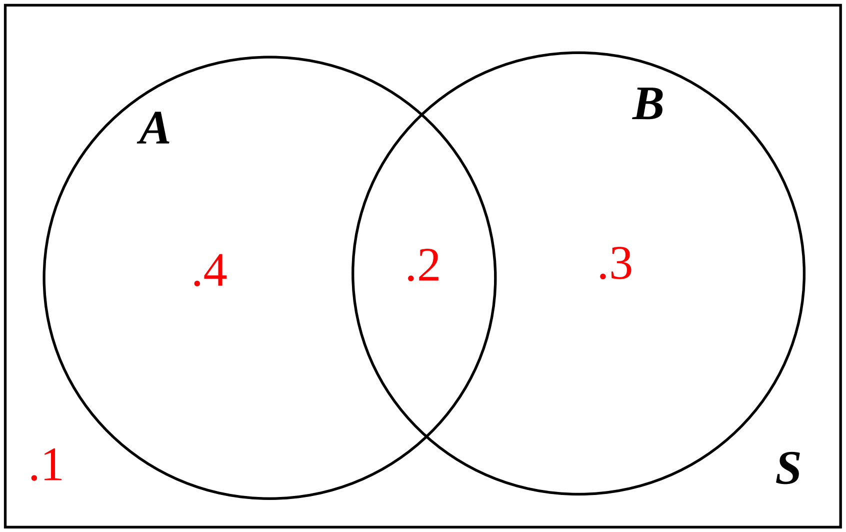
$P(J \mid F)$

$$\frac{2}{3}$$

If you have the option of using either the reduced sample space method or the conditional probability formula, it's usually easier to use the reduced sample space method.

2. $P(A) = .6$**,** $P(B) = .5$**, and** $P(A \cap B) = .2$**.** In this example, we must use the conditional probability formula.

a) Complete the probability diagram.



b) Find $P(A|B)$

$$\frac{.2}{.5} = \boxed{\frac{2}{5}}$$

$P(B|A)$

$$\frac{.2}{.6} = \boxed{\frac{1}{3}}$$

$P(A|\bar{B})$

$$\frac{.4}{.5} = \boxed{\frac{4}{5}}$$

$P(\bar{B}|A)$

$$\frac{.4}{.6} = \boxed{\frac{2}{3}}$$

3. Two cards are randomly drawn from a standard 52-card deck in succession without replacement. Reduced sample space is the way to go.

a) $P(\text{second card is a heart} \mid \text{the first card is a heart})$

$$\frac{12}{51} = \frac{4}{17}$$

b) $P(\text{second card is a heart} \mid \text{the first card is a club})$

$$\frac{13}{51}$$

4. A survey of North Harris students had the following results.

	Pepperoni	Sausage	Mushroom	Total
Freshman	**25**	**15**	**5**	**45**
Sophomore	**30**	**20**	**5**	**55**
Total	**55**	**35**	**10**	**100**

A student from the survey is *selected at random*. Reduced sample space.

a) $P(\text{freshman} \mid \text{pepperoni})$

$$\frac{25}{55} = \frac{5}{11}$$

b) $P(\text{pepperoni} \mid \text{freshman})$

$$\frac{25}{45} = \frac{5}{9}$$

c) $P(\text{sophomore} \mid \text{sausage})$

$$\frac{20}{35} = \frac{4}{7}$$

d) $P(\text{sausage} \mid \text{sophomore})$

$$\frac{20}{55} = \frac{4}{11}$$

<u>Product Rule:</u> Rearranging the conditional probability formula leads to the product rule.

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \textbf{ so}$$

$$P(A \cap B) = P(A) \cdot P(B|A)$$

Examples:

Two cards are drawn at random from a standard 52-card deck in succession without replacement. Find

$$P(\text{first is a heart and the second is a heart})$$

$$= P(\text{first is a heart}) \cdot P(\text{second is a heart}|\text{first is a heart})$$

$$= \frac{13}{52} \cdot \frac{12}{51} = \boxed{\frac{1}{17}}$$

$$P(\text{first is a club and the second is a heart})$$

$$= P(\text{first is a club}) \cdot P(\text{second is a heart}|\text{first is a club})$$

$$= \frac{13}{52} \cdot \frac{13}{51} = \boxed{\frac{13}{204}}$$

$$P(\text{first is a club and the second is a heart and the third is a heart})$$

$\frac{13}{52} \cdot \frac{13}{51} \cdot \frac{12}{50} = \frac{13}{850}$ The product rule works for more than just two events.

Independence and Dependence:

The events *E* and *F* are independent if any of the following three conditions are true.

$$P(E \mid F) = P(E)$$

$$P(F \mid E) = P(F)$$

Events that clearly have no affect on each other are considered to be independent. This includes flips of coins and rolls of dice.

$$P(E \cap F) = P(E) \cdot P(F)$$

Events that are not independent are dependent.

$P(E) = .5$**,** $P(F) = .6$**, and** $P(E \cap F) = .3$

$$P(E) \cdot P(F) = (.5)(.6) = .3 = P(E \cap F)$$

So E and F are independent.

$P(A) = .7$**,** $P(B) = .6$**, and** $P(A \cap B) = .4$

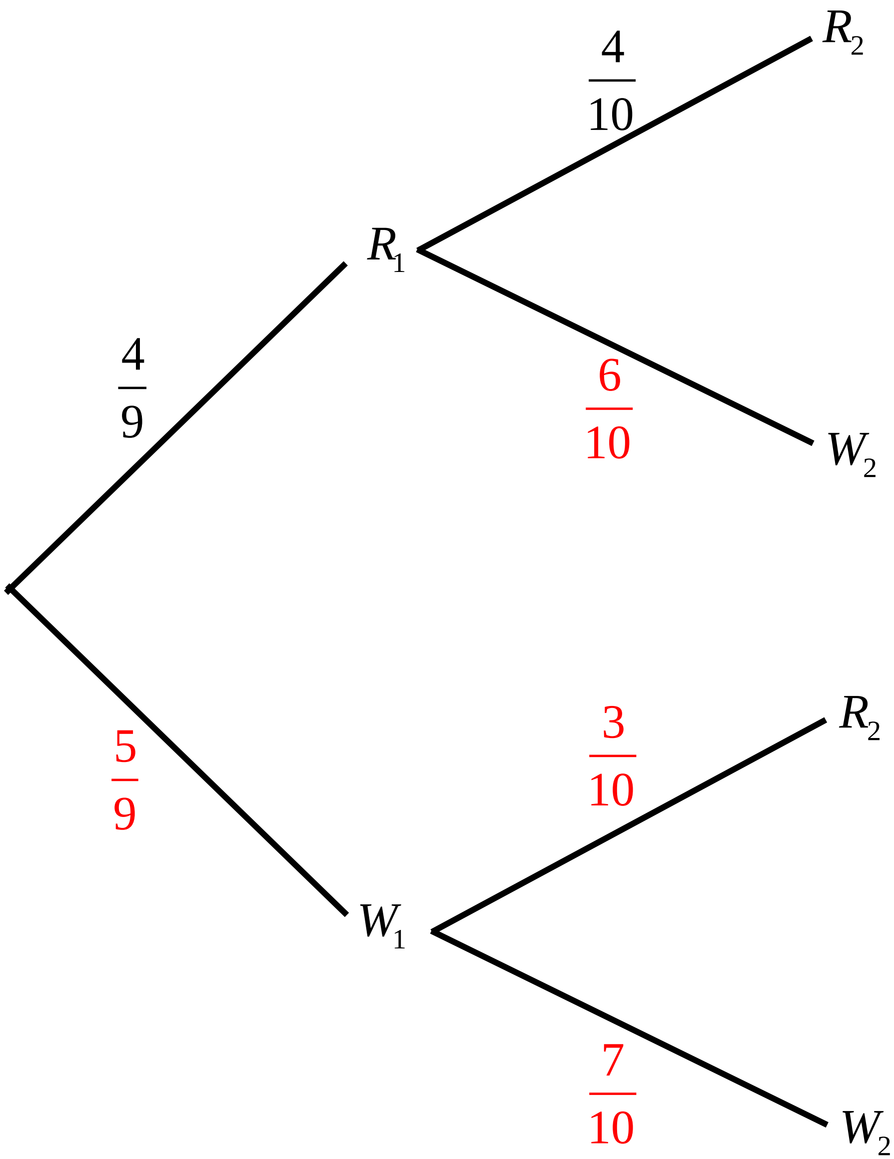
$$P(A) \cdot P(B) = (.7)(.6) = .42 \neq .4 = P(A \cap B)$$

So A and B are dependent.

<u>Probability Trees:</u>

They keep track of the probabilities and conditional probabilities in a multistage experiment.

Example: There are two bowls: bowl 1 has 4 red marbles and 5 white marbles, and bowl 2 has 3 red marbles and 6 white marbles. A marble is randomly selected from bowl 1 and transferred into bowl 2. Then a marble is randomly selected from bowl 2.



The sum of the probabilities or conditional probabilities originating from the same branch point must be 1 in a probability tree.

To find the probability of a sequence of events starting at the left and ending at a branch tip on the right, just multiply the probabilities and conditional probabilities along the path.

Find

$P(R_1 \cap R_2)$

$$\frac{4}{9}\cdot\frac{4}{10}=\boxed{\frac{8}{45}}$$

$P(R_2)$

$$\frac{4}{9}\cdot\frac{4}{10}+\frac{5}{9}\cdot\frac{3}{10}=\boxed{\frac{31}{90}}$$

$P(W_2)$

$$\frac{4}{9}\cdot\frac{6}{10}+\frac{5}{9}\cdot\frac{7}{10}=\boxed{\frac{59}{90}}$$

or

$$1-P(R_2)=1-\frac{31}{90}=\frac{59}{90}$$

$P(R_1 \mid R_2)$

$$=\frac{P(R_1 \cap R_2)}{P(R_2)}=\frac{\frac{8}{45}}{\frac{31}{90}}=\boxed{\frac{16}{31}}$$

$P(W_1 \mid R_2)$

$$=\frac{P(W_1 \cap R_2)}{P(R_2)}=\frac{\frac{5}{9}\cdot\frac{3}{10}}{\frac{31}{90}}=\frac{\frac{15}{90}}{\frac{31}{90}}=\boxed{\frac{15}{31}}$$

or

$$1-P(R_1 \mid R_2)=1-\frac{16}{31}=\frac{15}{31}$$

$P(R_1 \mid W_2)$

$$=\frac{P(R_1 \cap W_2)}{P(W_2)}=\frac{\frac{4}{9}\cdot\frac{6}{10}}{\frac{59}{90}}=\boxed{\frac{24}{59}}$$

$P(W_1 \mid W_2)$

$$=\frac{P(W_1 \cap W_2)}{P(W_2)}=\frac{\frac{5}{9}\cdot\frac{7}{10}}{\frac{59}{90}}=\boxed{\frac{35}{59}}$$

or

$$1-P(R_1 \mid W_2)=1-\frac{24}{59}=\frac{35}{59}$$