Arsenio Says, Show Me The Digits!

1. a) From the digits 0, 1, 2, 3, 4, 5, 6, how many four-digit numbers with distinct digits can be constructed? *{0463 is not a four-digit number!}*

b) Of these, how many are even?

c) How many different four-digit numbers can be formed by arranging the digits 2, 1, 1, and 6?

Being Wholly Positive About The Number Of Divisors

2. How many distinct positive whole number divisors are there of the integer $30^4 = 810{,}000$?

{Hint: $30^4 = 2^4 \cdot 3^4 \cdot 5^4$*, so every divisor is uniquely determined by*

Number of factors of 2	**Number of factors of 3**	**Number of factors of 5**

}

Fair-minded Santa.

3. a) In how many ways can 9 different toys be divided evenly among three children?

{Hint: The distribution of toys boils down to

Which 3 toys for child #1	**Which 3 toys for child #2**	**Which 3 toys for child #3**

}

b) In how many ways can 9 identical toys be divided evenly among three children?

Boys Are Icky; No Girls Are Icky.

4. Three boys and 3 girls will sit together in a row.

a) How many different ways can they sit together without restrictions?

b) How many different ways can they sit together if the genders must sit together?

c) How many different ways can they sit together if only the boys must sit together?

d) How many different ways can they sit together if no two of the same gender can sit together?

You've Seen One Painting, You've Seen Them All.

5. An art collection on auction consisted of 4 Dalis, 5 Van Goghs, and 6 Picassos, and at the art auction were 5 art collectors. The society page reporter only observed the number of Dalis, Van Goghs, and Picassos acquired by each collector.

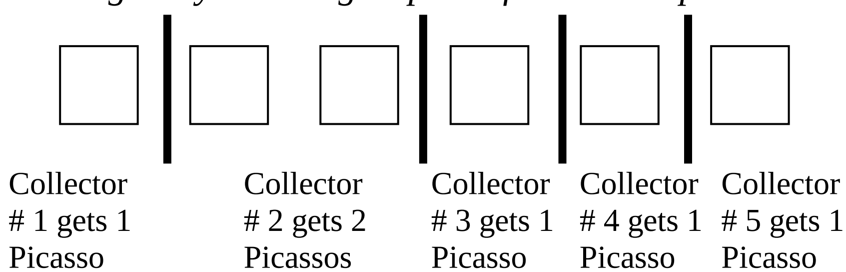
a) How many different results could she have recorded for the sale of the Dalis if all were sold?

b) How many different results could she have recorded for the sale of the Van Goghs if all were sold?

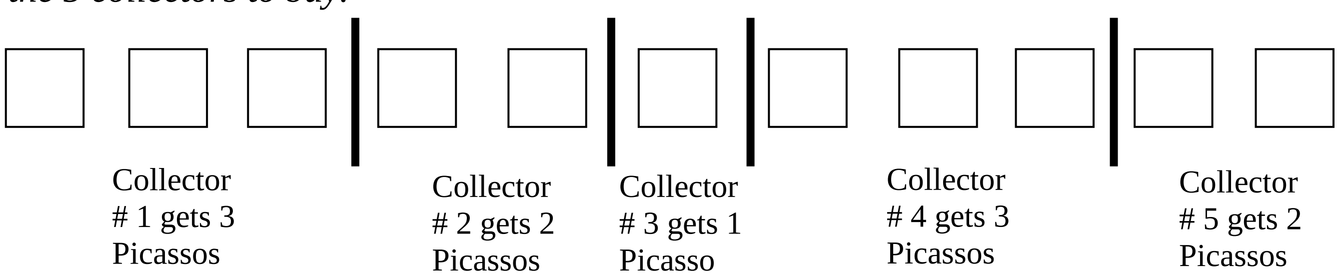
c) How many different results could she have recorded for the sale of the Picassos if all were sold?

d) How many different results could she have recorded for the sale of all 15 paintings if all were sold?

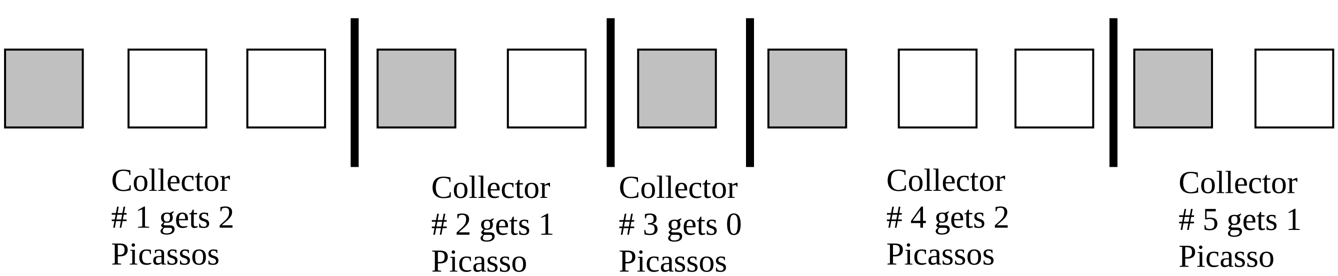
{Hint: If we assume that each collector buys at least one Picasso then we'll decide how many each collector gets by choosing 4 spaces from the 5 spaces between the 6 Picassos:



So if each must buy at least one, there are $_5C_4 = 5$ *different ways that the 6 Picassos could have been sold to the 5 collectors. To allow for the possibility that one or more collectors didn't buy any Picassos, we'll pretend that there are actually 11 Picassos for the 5 collectors to buy.*



From the 10 spaces available, we'll select 4. If we subtract 1 from each number of Picassos assigned to each collector, we'll have a way that the collectors could buy all 6 Picassos even if some don't buy any.



}

Bob And Carol And Ted And Alice And …

6. Three married couples have bought six seats in a row for a performance of a musical comedy.

a) In how many different ways can they be seated?

b) In how many different ways can they be seated if each couple must sit together with the husband to the left of his wife?

c) In how many different ways can they be seated if each couple must sit together?

d) In how many different ways can they be seated if all the men must sit together and all the women must sit together?

Fancy Dealing.

7. How many different ways can you select 13 cards out of a standard 52 card deck so that the 13 cards selected include at least 3 cards from each suit?

{Hint: If you have at least 3 cards of each of the four suits, that gives you 12 cards. You just need one more card.}

If you count it using

$_{13}C_3$	$_{13}C_3$	$_{13}C_3$	$_{13}C_3$	40
# of ways to choose 3 hearts	# of ways to choose 3 diamonds	# of ways to choose 3 spades	# of ways to choose 3 clubs	# of ways to choose the 13^{th} card

You will over count. Here's why:
Suppose that one time you choose 1, 2, and 3 of hearts, 1, 2, and 3 of diamonds, 1,2, and 3 of spades, 1, 2, and 3 of clubs and your 13^{th} card is the 4 of hearts, and the next time you choose 1, 2, and 4 of hearts, 1, 2, and 3 of diamonds, 1,2, and 3 of spades, 1, 2, and 3 of clubs and your 13^{th} card is the 3 of hearts. Then the two selections are the same, but they are counted as two different selections.

Instead, try the approach

Which suit will have 4 cards?	# of ways to choose 4 of this kind	# of ways to choose 3 of the next kind	# of ways to choose 3 of the next kind	# of ways to choose 3 of the last kind

Ups and Downs With And Without Nine Lives.

8. a) An elevator starts at the basement with 8 people(not including the elevator operator) and discharges them all by the time it reaches the 6th floor. In how many ways could the operator record the number of people leaving the elevator on each of the 6 floors?

b) If the same elevator also has 10 cats, in how many ways could the operator record the number of cats leaving the elevator on each of the 6 floors?

c) In how many ways could the operator record the number of people and the number of cats leaving the elevator on each of the 6 floors?
{Hint: See the hint for #5.}

Don't Spend It All In One Place.

9. We have $20,000 dollars that must be invested among 4 possible opportunities. Each investment must be a whole number multiple of $1,000, and there are minimal investments that must be made. The minimal investments are 2, 2, 3, and 4 thousand dollars, respectively. How many different investment strategies are available?

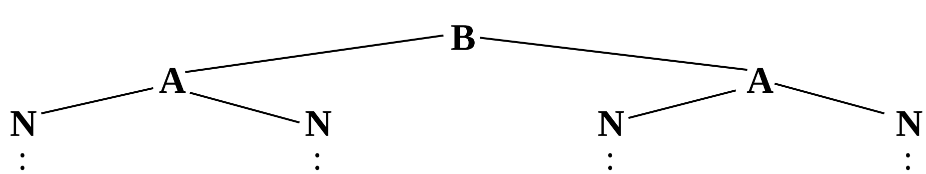
{Hint: See the hint for #5.}

Hopefully, You'll Have A Lot Of Interest In These Banks.

10. Determine the number of different paths for spelling the word BANK:

K			
N	**K**		
A	**N**	**K**	
B	**A**	**N**	**K**

{Hint: The letters actually form a tree diagram:



}

It's All In The Name.

11. a) Explain why in a group of 677 people with names spelled from the letters A-Z, at least two people share first and last names beginning with the same letters. For example, the names could be Chris Jones and Charles Jackson.

{Hint: How many different ways are there for the beginning letters of a person's first and last names?

# of choices for the first letter of the first name	*# of choices for the first letter of the last name*

}

b) What is the fewest number of people needed to guarantee that at least two people share first, middle, and last names beginning with the same letters?(Assume that everyone has first , middle, and last names.) For example, the names could be Chris Allen Jones and Charles Arnold Jackson.

c) What is the fewest number of people needed to guarantee that at least three people share first and last names beginning with the same letters?

d) What is the fewest number of people needed to guarantee that at least three people share first, middle, and last names beginning with the same letters?(Assume that everyone has first , middle, and last names.)

It's A Monthly Thing.

12. a) What is the smallest number of people in a group that will guarantee that at least two of the people were born in the same month?

b) What is the smallest number of people in a group that will guarantee that at least three of the people were born in the same month?

Texas Hold'em.

13. In this problem, we'll determine the number of possible particular 5-card poker hands.

Here is a possible decision process for the 5-card poker hand with one pair

13	${}_4C_2$	${}_{12}C_3$	4	4	4
Which kind of pair?	Which two cards of this kind?	Which 3 other kinds?	Which one of the first kind?	Which one of the second kind?	Which one of the third kind?

So there are $13 \cdot {}_4C_2 \cdot {}_{12}C_3 \cdot 4 \cdot 4 \cdot 4 = 1{,}098{,}240$ different two-of-a-kind 5-card poker hands.

a) See if you can do the same thing to find the number of different three-of-a-kind hands:

Which kind of three-of-a-kind?	Which three cards of this kind?	Which 2 other kinds?	Which one of the first kind?	Which one of the second kind?

b) See if you can do the same thing to find the number of different four-of-a-kind hands:

Which kind of four-of-a-kind?	Which other kinds?	Which one of the other kind?

The number of different flushes, i.e. five cards of the same suit, but not in order

First we'll count the number of different hands with 5 cards of the same suit:

4	${}_{13}C_5$
Which suit?	Which 5 cards?

Then we'll subtract the number of hands with 5 cards of the same suit that are in order(these would be straight flushes):

4	10
Which suit?	Which kind of card starts the straight flush?

So we get $4 \cdot {}_{13}C_5 - 4 \cdot 10 = 5{,}108$ different 5-card poker hands which are flushes.

c) See if you can do something similar to find the number of straights, i.e. 5 cards in a row, but not all of the same suit.

First we'll count the number of different hands with 5 cards in a row:

Which kind of card starts the straight?	Which suit for the first card?	Which suit for the second card?	Which suit for the third card?	Which suit for the fourth card?	Which suit for the fifth card?

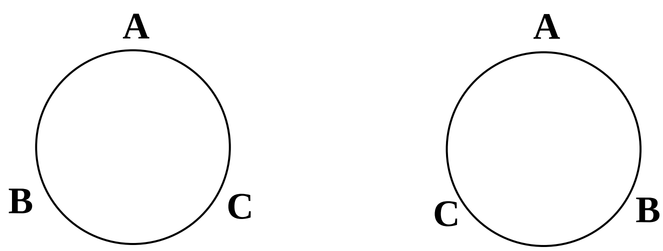
Then we'll subtract the number of 5-card hands in order of the same suit(straight flushes):

Red Or White, It's Your Joyce.

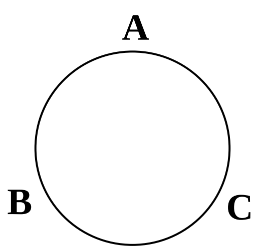
14. To win a math contest, Joyce must determine how many marbles are in a box. She is told that there are 3 identical red marbles and some number of identical white marbles in the box. She is also told that there are 35 distinguishable permutations of the marbles. So how many marbles are in the box?

{Hint: The number of distinguishable permutations is $\frac{(R+W)!}{R!\cdot W!}$*, and we know that* $R=3$*.}*

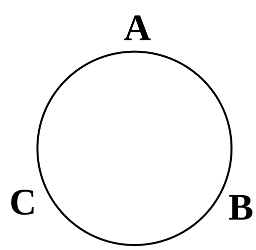
You have learned that the number of permutations of *n* distinct objects is *n*!. For instance if you wanted to seat three people along one side of a rectangular table, the number of possible arrangements is 3!. However, if the three people are to be seated around a circular table, the number of possible arrangements is only 2!. Let's see why: If the people are labeled A, B, and C, the two arrangements look like the following:



At first, it might seem that there should be 3! = 6 different arrangements, like the following:



(1)



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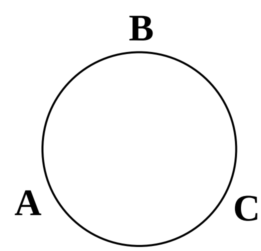
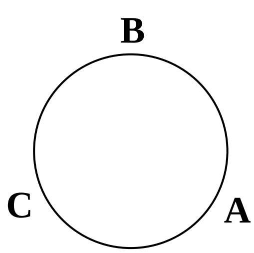
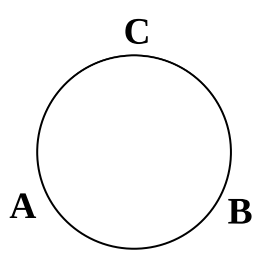


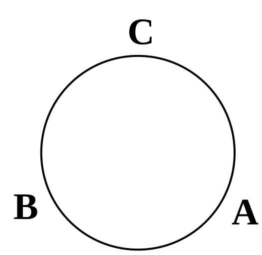
(3)



(4)



(5)



(6)

But, if you look closely, you'll see that arrangements (1), (4), and (5) are identical, each is just a rotation of the other. The same is true of (2), (3), and (6).

Knights Of The Circular Table And The Venerable Bead.

15. a) Find a formula for the number of different ways that *n* people(or objects) can be seated(or placed) around a circular table.

{Hint: Start with n!, but divide it by the number of rotations that can be made that generate equivalent arrangements.}

b) Use the previous formula to find the number of different arrangements of 12 people around a circular table.

c) Use the previous formula to find the number of different necklaces that use 10 different colored beads.

d) Modify the previous formula to find the number of different necklaces that use 20 beads with 5 red, 4 blue, 8 green, and 3 yellow.

{Hint: Similarly modify the formula for permutations of non-distinguishable objects.}

Multiples Of Multiples.

16. a) How many of the first 1,000 counting numbers are multiples of 2 or multiples of 5?

{Hint: $\begin{aligned} n(\text{multiples of 2 or multiples of 5}) &= n(\text{multiples of 2}) + n(\text{multiples of 5}) \\ &\quad - n(\text{multiples of 2 and 5}) \end{aligned}$ *.}*

b) How many of the first 10,003 counting numbers are multiples of 2 or multiples of 3?

Don't Get Punched Out At The Motel.

17. A national motel chain has replaced the key lock for each room with a key card system. A door is unlocked by inserting a plastic card into a slot above the door knob. Each key's unique identity is determined by a grid of 63 cells, each of which is either solid or punched.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	
19	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**Room**
28	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	#__________
37	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**	
46	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	
55	**56**	**57**	**58**	**59**	**60**	**61**	**62**	**63**	

a) Determine the number of different key cards possible.

b) How many are possible if each key card must have at least one punched cell?

It's As Easy As ABCDEFG.

18. Find the number of permutations of ABCDEFG that contain the following:

a) the sequence ABC

{Hint: They would look like one of the following:

A	*B*	*C*				

	A	*B*	*C*			

		A	*B*	*C*		

			A	*B*	*C*	

				A	*B*	*C*

.}

b) the sequences AB, CD, and EF, but not necessarily in this order.

{Hint: Treat each pair of letters as a single unit, and decide the position of G. For example,

AB	*CD*	*EF*	*G*

AB	*CD*	*G*	*EF*

AB	*G*	*CD*	*EF*

G	*AB*	*CD*	*EF*

.}

c) the sequences AB, BC, and EF, but not necessarily in this order.

Man Have We Got A Lot Of Different Pizzas.

19. A pizza parlor offers four sizes of pizza(small, medium, large, and colossus), two types of crust(thick and thin), and 14 different toppings.

a) If you must choose a size, a crust, and at least 1 topping, and you can't duplicate a topping, how many different pizzas can be made?

b) How many if double toppings are allowed?

Too Many Officers And Not Enough Enlisted.

20. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people(A, B, C, D, E, F, G, H, I, J). How many different choices of officers are possible if

a) there are no restrictions?

b) A and B will not serve together?

{Hint: Some selections will have only B, some only A, and some won't have either.}

1	8	7
B is President	Treasurer	Secretary

8	1	7
President	A is Treasurer	Secretary

c) C and D will serve together or not at all?

{Hint: Some selections will have C and D, and some won't have either.}

1	1	8
C is President	D is Treasurer	Secretary

d) E must be an officer?

{Hint: E has to be one of the officers selected.}

9	8	1
President	Treasurer	E is Secretary

e) F will only serve if she is president?

{Hint: Some selections will have F as president, and some won't have F as an officer.}

1	9	8
F is President	Treasurer	Secretary

Even You Can Choose Two.

21. In how many different ways can you select two distinct integers from the set $\{1,2,3,\ldots,100\}$ so that their sum is even?

{Hint: What kinds of numbers will produce even sums? $even + even = ?$, $odd + odd = ?$, $even + odd = ?$ *}*

Congratulations Mr. and Mrs. Zeta.

22. Mr. and Mrs. Zeta want to name their baby so that its monogram(first, middle, and last initials) will be in alphabetical order with no letters repeated. How many such monograms are possible?

Shake It Like You Mean It.

23. a) Twenty people are at a party. If everyone at the party shakes the hand of everyone else at the party, determine the total number of handshakes.

b) Ten married couples are having a party. If each person at the party shakes the hand of everyone else except his/her spouse, determine the number of handshakes at the party.

Don't Lose Your Marbles!

24. An urn contains 5 red, 6 blue, and 8 green marbles. If a set of 3 marbles is randomly selected, without replacement,

a) what is the probability that all the marbles will be of the same color?

b) what is the probability that the marbles will be of different colors?

c) Answer parts a) and b) if each time a marble is selected, its color is recorded and then it's put back into the urn.

Imelda Returns.

25. A closet contains 10 pairs of shoes. If 8 shoes are selected, without replacement, how many ways can there be

a) no complete pair?

Hint: Here are the ten pairs of shoes:

L_1R_1	L_2R_2	L_3R_3	L_4R_4	L_5R_5	L_6R_6	L_7R_7	L_8R_8	L_9R_9	$L_{10}R_{10}$

${}_{10}C_8$	2	2	$\cdots$	2
Which 8 pairs?	**Shoe from 1st pair selected**	**Shoe from 2nd pair selected**	$\cdots$	**Shoe from 8th pair selected**

b) exactly one complete pair?

Bob And Carol And Ted And Alice And … The Saga Continues.

26. Six married couples are standing in a room.

a) If two people are chosen at random, find the probability that they are married.

b) If two people are chosen at random, find the probability that one is male and one is female.

c) If four people are chosen at random, find the probability that two married couples are chosen.

d) If four people are chosen at random, find the probability that no married couple is chosen.

e) If the 12 people are divided into 6 pairs, find the probability that each pair is married.

f) If the 12 people are divided into 6 pairs, find the probability that each pair contains a male and a female.

It's All Ancient Greek To Me.

27. You have wandered by accident into a class in ancient Greek. A ten-question multiple choice test is handed out, with each answer to be chosen from four possibilities. If you randomly guess the answers, what is the probability that you will get all the answers right?

There is an important principle in probability that generalizes the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to more than two events, and it's called the Inclusion-exclusion Principle. For three events, we can derive it from the formula for two events, assuming that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cap (A \cap C) = A \cap B \cap C$:

$$\begin{aligned} P(A \cup B \cup C) = P(A \cup (B \cup C)) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) \\ &\quad + P(A \cap B \cap C) \end{aligned}.$$

In short, $$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) \\ &\quad + P(A \cap B \cap C) \end{aligned}.$$ The pattern in general is to add the single event probabilities, subtract the double event probabilities, add the triple event probabilities,…. So for example with four events the formula is

$$P(A\cup B\cup C\cup D)=P(A)+P(B)+P(C)+P(D)-P(A\cap B)-P(A\cap C)-P(A\cap D)-P(B\cap C)$$
$$-P(B\cap D)-P(C\cap D)+P(A\cap B\cap C)+P(A\cap B\cap D)+P(A\cap C\cap D)$$
$$+P(B\cap C\cap D)-P(A\cap B\cap C\cap D)$$

You may use the Inclusion-exclusion Principle in solving the next two problems.

I'm Missing A Suit, But Don't Call The Dry Cleaner.

28. Find the probability that 13 randomly selected cards from a standard 52 card deck will be missing at least one of the four suits.

{Hint: Let C be the event that the 13 cards are missing clubs, D be the event that the 13 cards are missing diamonds, H be the event that the 13 cards are missing hearts, and S be the event that the 13 cards are missing spades.

$$P(C\cup D\cup H\cup S)=P(C)+P(D)+P(H)+P(S)-P(C\cap D)-P(C\cap H)-P(C\cap S)$$
$$-P(D\cap H)-P(D\cap S)-P(H\cap S)+P(C\cap D\cap H)+P(C\cap D\cap S)$$
$$+P(D\cap H\cap S)+P(C\cap H\cap S)-P(C\cap D\cap H\cap S)\}$$

The Mad Hat-Checker.

29. A hat-checker in a theater mixed up all five of his checks and decided to hand out all of the hats at random at the end of the show. What is the probability that not a single person received his/her hat back?

{Hint: Let A be the event that the first owner gets his hat back, B the event that the second owner gets his hat back, with events C, D, and E defined similarly. The probability that we want to calculate is $1-P(A\cup B\cup C\cup D\cup E)$, so use the Inclusion-exclusion Principle to do it.}

Roll The Bones Differently.

30. a) If four fair dice are thrown, what is the probability that the four numbers turning up will all be different?

{Hint: Each of the different ways the dice could turn up different would correspond to a permutation of size 4 of the numbers 1-6.}

b) If six fair dice are thrown, then what is the probability that at least one number will repeat?

Eleven The Hard Way.

31. a) A fair coin is tossed 14 times, find the probability of tossing at least 11 consecutive tails.

{Hint: Here are the different ways of having exactly 11 consecutive tails:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
T	T	T	T	T	T	T	T	T	T	T	H	H or T	H or T
H	T	T	T	T	T	T	T	T	T	T	T	H	H or T
H or T	H	T	T	T	T	T	T	T	T	T	T	T	H
H or T	H or T	H	T	T	T	T	T	T	T	T	T	T	T

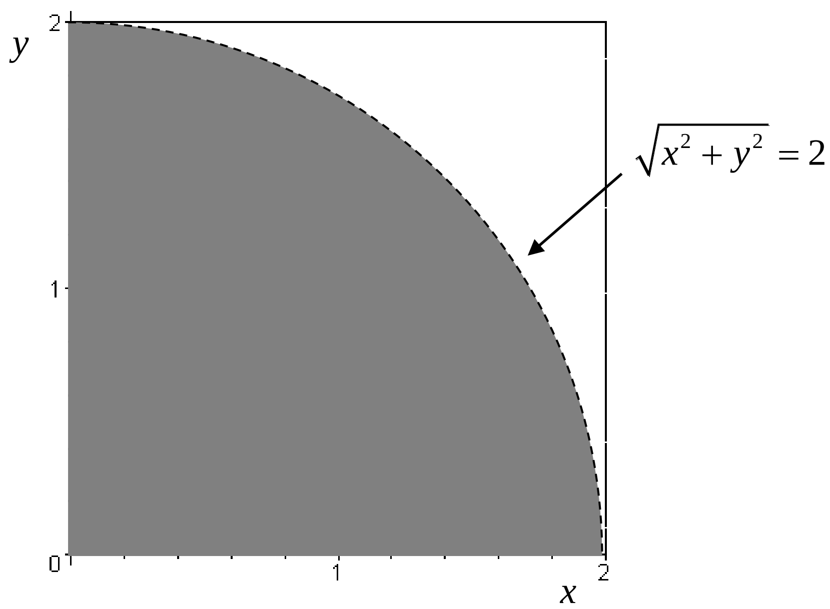
So you get $2^2 + 2 + 2 + 2^2 = 12$*. Do the same for exactly 12, exactly 13, and exactly 14.}*

b) A fair coin is tossed 15 times, find the probability of tossing at least 11 consecutive tails.

It's The Length Of The Hypotenuse That Counts.

32. a) If x and y are numbers randomly chosen between 0 and 2, what is the probability that the hypotenuse of a right triangle with legs of x and y will have length less than 2?

{Hint:



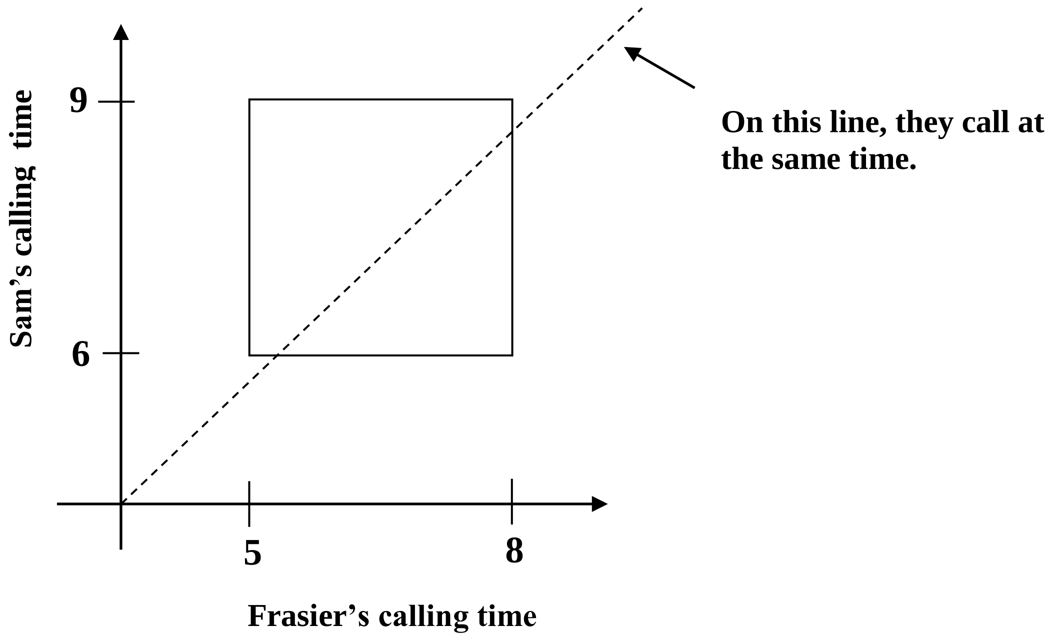
}

b) Do the same, but for the hypotenuse length being less than 1.

Where Everybody Knows Your Name.

33. Frasier and Sam both plan to call Diane tonight to ask her for a date. Frasier plans to call between 5:00 PM and 8:00 PM while Sam plans to call between 6:00 PM and 9:00 PM. If they pick their actual calling times at random, what is the probability that Frasier calls before Sam?

{Hint:



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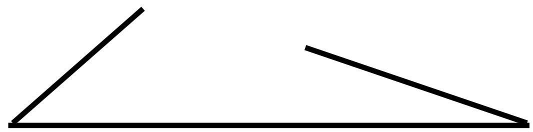
We're All Winners Here.

34. Three runners compete in a race. The probability that A will win the race is twice the probability that B will win. The probability that B will win is twice the probability that C will win. What is the exact probability that A will win the race?

Just Stick To It.

35. a) A box contains 5 sticks measuring 15, 30, 40, 60, and 90 centimeters in length. If three of the sticks are randomly chosen, what is the probability that they can be arranged to form a triangle?

{Hint: In order for the three sticks to form a triangle, the length of the longest stick must be less than the sum of the lengths of the other two sticks.



Longest stick	Medium stick	Small stick
90	60	40
60	40	30
⋮	⋮	⋮

}

b) Do the same, but the lengths are 15, 30, 40, 50, and 80.

Another Marble Question Out Of The Blue.

36. A box contains less than 20 marbles. If you randomly select two marbles without replacement,

a) you have a $\frac{2}{3}$ chance of getting two blue marbles. How many blue marbles are in the box?

b) you have a 50% chance of getting two blue marbles. How many blue marbles are in the box?

A Patriotic Marble Problem.

37. A bag contains marbles which are colored red, white, or blue. The probability of drawing a red marble is $\frac{1}{6}$, and the probability of drawing a white marble is $\frac{1}{3}$.

a) What is the probability of drawing a blue marble?

b) What is the smallest number of marbles that could be in the bag?

c) If the bag contains four red marbles and eight white marbles, how many blue marbles does it contain?

Hey, What's Your Sign?

38. There are 12 signs of the zodiac. Assuming that a person is just as likely to be born under one sign as another, what is the probability that in a group of 5 people at least two of them

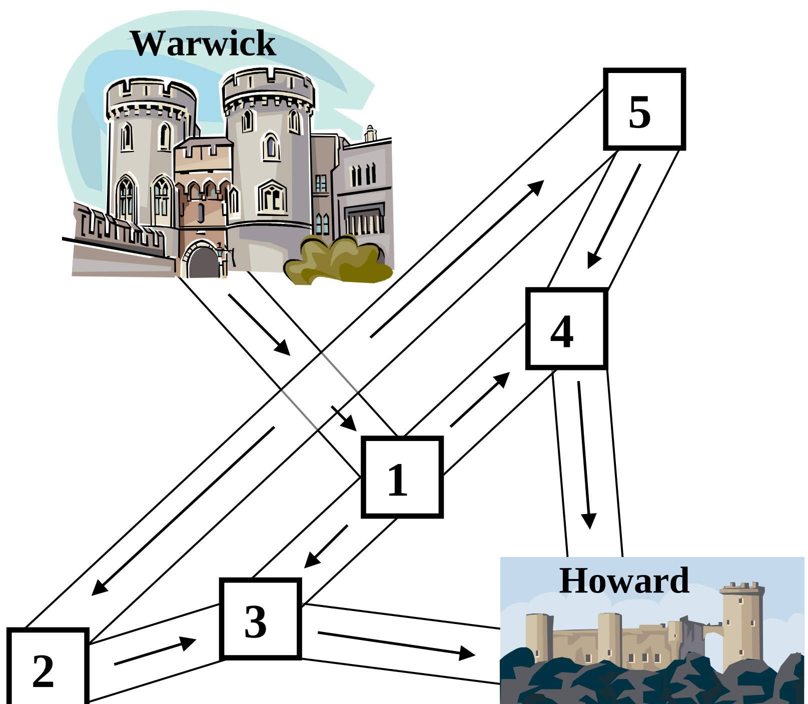
a) have the same sign?

{Hint: The opposite of at least two have the same sign is that all 5 have different signs.}

b) are Aries?

Escape from Castle Warwick.

39. You're trapped at Castle Warwick. The only escape is to reach Castle Howard through a system of canals. The problem is that the system of canals has five gatehouses, each run by a cranky gatekeeper who shows up for work about half the time. So the probability that a gate is open on a given day is one-half. The arrows show the way the water flows through the canals and indicate the only direction of travel. What is the probability that a water route from Castle Warwick to Castle Howard is open so that you can escape?



{Hint: There are $2^5 = 32$ *different states of the five gate houses. Here are the ways you can't get to Castle Howard*

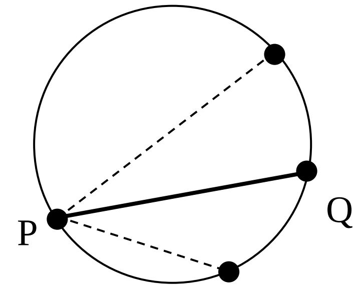
1	2	3	4	5
open or closed	open or closed	closed	closed	open or closed
closed	closed	open	closed	open or closed
closed	open or closed	closed	open	closed
closed	closed	open	open	closed

}

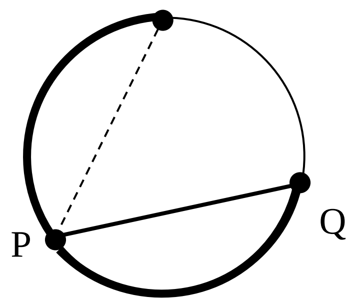
Raiders Of The Lost Arc.

40. A line segment PQ goes across a circle of radius 3.

a) Suppose that the shorter distance around the circle from P to Q is 2π. What is the probability that a second line segment drawn at random from P to another point on the circle will be shorter than PQ?



Hint: The circumference of the circle is 6π. *The length of portion of the circumference where we can choose a point R so that* $\overline{PR}$ *is shorter than* $\overline{PQ}$ *is* 4π.



b) Suppose that the distance around the circle from P to Q is 3π. What is the probability that a second line segment drawn at random from P to another point on the circle will be shorter than PQ?

I Repeat; Don't Lose Your Marbles!

41. A bowl initially contains 5 white and 7 black marbles. Each time a marble is selected, its color is noted and it is replaced in the bowl along with 2 other marbles of the same color.

a) Find the probability that the first two marbles selected are black and the next two white.

b) Find the probability that of the first 4 marbles selected, exactly two are black.

No Fair, Two Heads Are Better Than One.

42. A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random; when he flips it, it shows heads.

a) What is the probability that it is the fair coin?

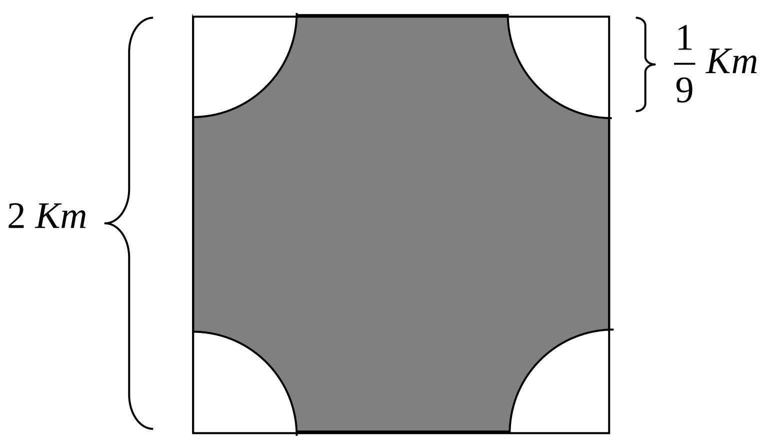
b) Suppose he flips the same coin a second time and again it shows heads. Now what is the probability that it is the fair coin?

c) Suppose he flips the same coin a third time and it show tails. Now what is the probability that it is the fair coin?

Why Would Anyone Jump Out Of A Perfectly Good Airplane?

43. A parachutist will jump from an airplane and land in a square field that is 2 kilometers on each side. In each corner of the field there is a group of trees. The parachutist's ropes will get tangled in a tree if he lands within $\frac{1}{9}$ kilometer of a corner. What is the probability that the parachutist will land in the field without getting caught in a tree?

{Hint: The parachutist must land inside the gray region of the square.



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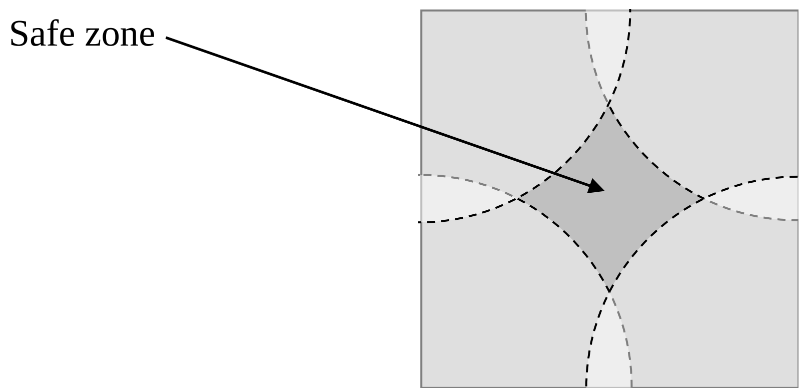
No Matter What You Do, You're Gonna Get Tangled Up.

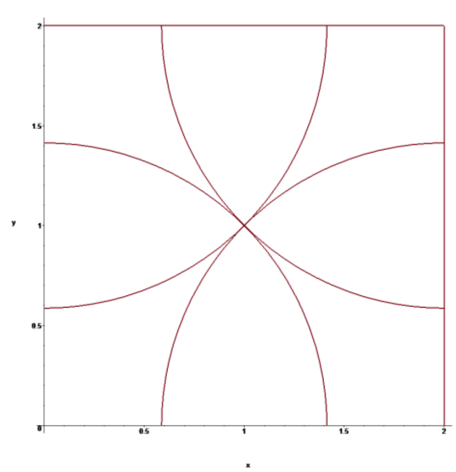
44. Refer to the previous parachute problem. Suppose the parachutist gets caught in a corner tree if he lands within x kilometers of a corner. Find the smallest value of x so that the probability of getting caught is 1?

{Hint: You might think that the answer is $\frac{2}{\sqrt{\pi}}$ *because* $\pi \cdot \left(\frac{2}{\sqrt{\pi}}\right)^2 = 4$*, but here's what the picture looks like with this radius:*



Notice the overlap of the tree regions.}

For the probability of getting caught in a corner tree to be 1 and x being the smallest, the picture would have to look like this.



Bowling For Marbles.

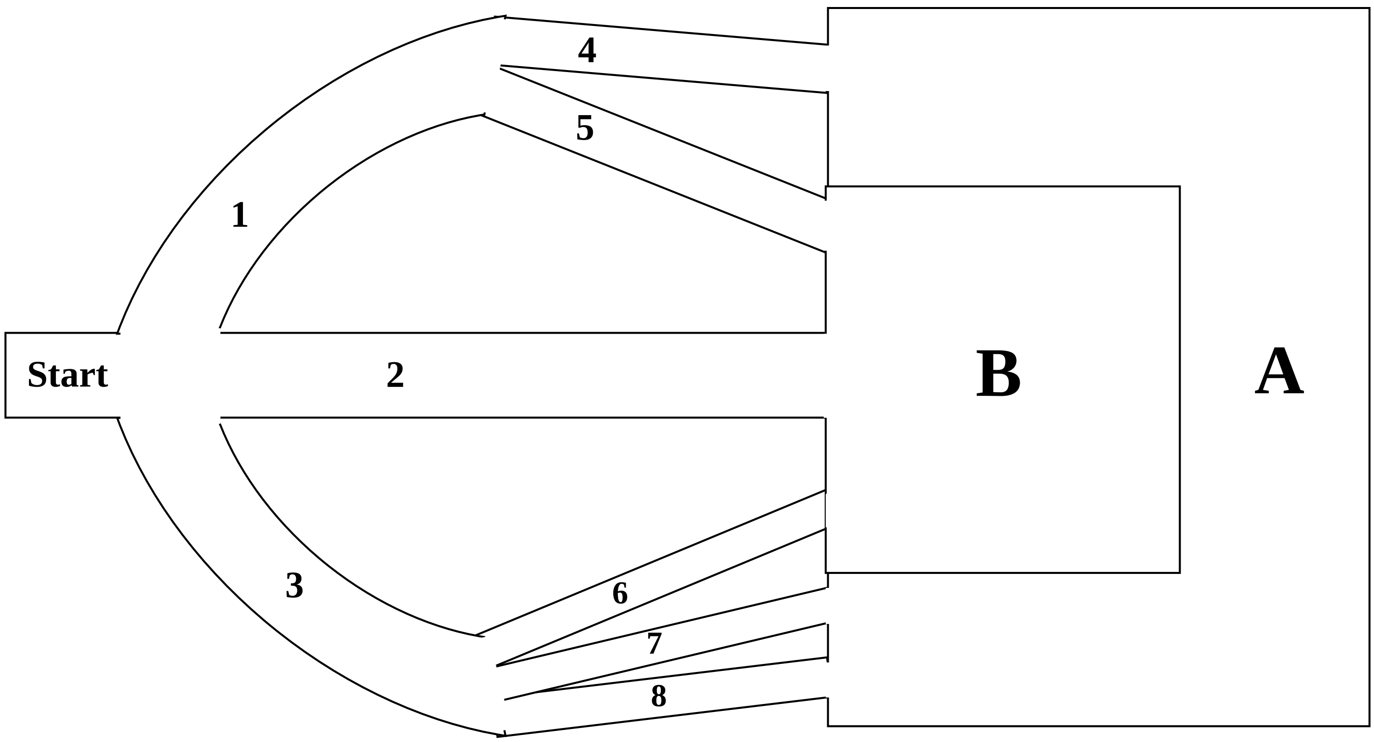
45. Bowl I contains 2 white and 4 red marbles, whereas bowl II contains 1 white and 1 red marble. A marble is randomly chosen from bowl I and transferred into bowl II, and then a marble is randomly selected from bowl II.

a) What is the probability that the marble selected from bowl II is white?

b) What is the probability that the transferred marble was white, given that a white marble is selected from bowl II?

The Worst You Get Is A *B*.

46. You enter the maze at the start and you choose the paths randomly moving from left to right until you arrive in either room A or room B.



a) Find the probability that you end up in room A.

b) Find the probability that you end up in room B.

c) Find the probability that path 3 was selected given that you ended up in room A.

d) Find the probability that path 3 was selected given that you ended up in room B.

e) Find the probability that path 2 was selected given that you ended up in room A.

{Hint: The maze path is almost a probability tree.}

It's All There In Black And White.

47. If two marbles are removed at random without replacement from a bag containing black and white marbles, the probability that they are both white is $\frac{1}{3}$. If three are removed at random without replacement, the probability that they are all white is $\frac{1}{6}$. How many marbles of each color are in the bag?

{Hint: $\frac{W}{W+B}\cdot\frac{W-1}{W+B-1}=\frac{1}{3}$ *and* $\frac{W}{W+B}\cdot\frac{W-1}{W+B-1}\cdot\frac{W-2}{W+B-2}=\frac{1}{6}$. *Substitute the first equation into the second one, and then substitute the result back into the first equation.}*

Kermit Asked: What's On The Other Side?

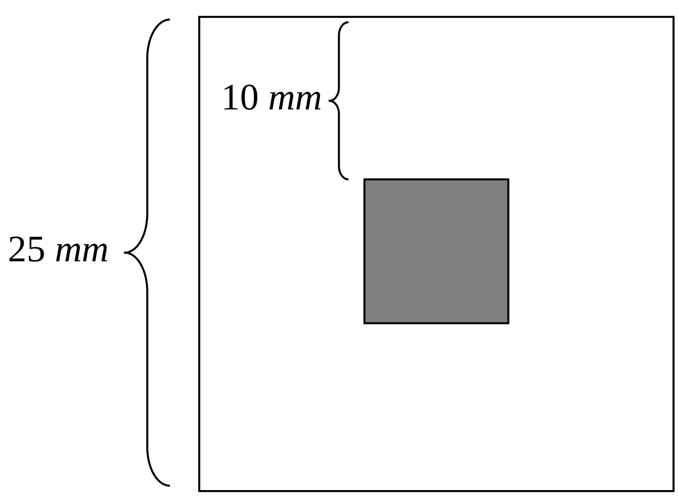
48. Three cards - one is red on both sides, another is black on both sides, and the other is red on one side and black on the other side – are placed into a hat. You randomly pull out one card and look at just one side of it. It is red. What is the probability that the card you hold is red on both sides?

{Hint: An equally likely sample space for this experiment is $S=\{\underbrace{(r_1,r_2),(r_2,r_1)}_{card\,1},\underbrace{(b_1,b_2),(b_2,b_1)}_{card\,2},\underbrace{(r_1,b_2),(b_2,r_1)}_{card\,3}\}$, *where each ordered pair represents what color is on the one side you look at and what color is on the other side.}*

How Fair Is The Fair?

49. At a state fair a game is played by tossing a coin of radius 10 millimeters onto a large table ruled into congruent squares each with side measure of 25 millimeters. If the coin lands entirely within some square, the player wins a prize. If the coin touches or crosses the edge of any square, the player loses. Assuming that the coin lands somewhere on the table, what's the probability that the player wins?

{Hint: The center of the coin must land inside the gray region of the square.



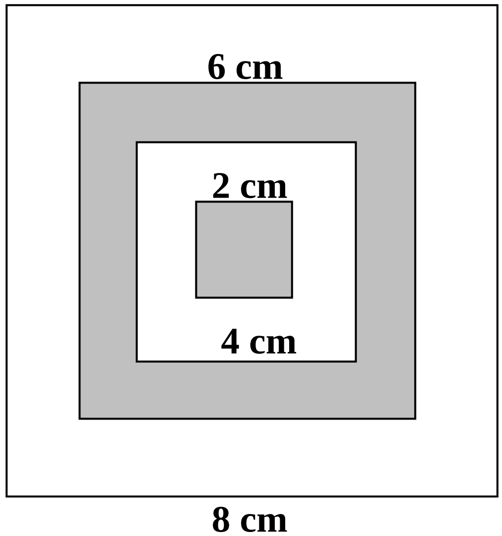
}

Let’s At Least Make It 50-50.

50. Refer to the previous problem. What should be the radius of the coin so that the probability of winning is .5?

You’re Just Throwing Darts.

51. If a dart hits the square target at random, what is the probability that it will hit in a shaded region?



{Hint: Think about area.}

Will It Be A Man Or A Woman?

52. One student in a class of men and women is to be chosen to represent the class. Each student is equally likely to be chosen, and the probability that a man is chosen is $\frac{2}{3}$ the probability that a woman is chosen. What is the ratio of the number of men to the total number of men and women?

{Hint: $\frac{M}{M+W}=\frac{\frac{2}{3}W}{M+W}$ *which implies that* $M=\frac{2}{3}W$*, so substitute this into the right side of the equation.}*

Oh, The French.

53. The French nobleman Antoine Gombauld, the Chevalier de Mere, was a famous 17^{th}-century gambler. He loved dice games. One of his favorites was betting that a 6 would appear at least once in four rolls of a die. He eventually became bored with this game, and came up with a new one. In the new game, he bet that there would be at least one pair of 6's in 24 rolls of a pair of dice. He soon noticed that he was not winning as much with the new game. In 1654, Gombauld wrote a letter to the French mathematician Blaise Pascal, who in turn mentioned the problem to Pierre de Fermat. The two mathematicians solved the mystery. See if you can too by working out the following parts. This exchange between gambler and mathematician is said to be the birth of the study of probability. In a letter to Fermat referring to Gombauld, Pascal wrote:
He is very intelligent, but he is not a mathematician: this as you know is a great defect.

a) Find the probability of getting at least one 6 in four rolls of a fair die.

{Hint: $P(\text{at least one 6 in four rolls}) = 1 - P(\text{no 6's in four rolls})$.*}*

b) Find the probability of getting at least one double 6 when rolling two fair dice 24 times.

{Hint: The probability of rolling a double 6 with a pair of fair dice is $\frac{1}{36}$*, and*

$P(\text{at least one double 6 in 24 rolls}) = 1 - P(\text{no double 6's in 24 rolls})$.*}*

c) What is the fewest number of rolls so that the probability of getting at least one double 6 is greater than $\frac{1}{2}$?

Tails You Lose.

54. Three fair coins are tossed and, after each toss, those coins that come up tails are removed, and the remaining coins tossed again. What is the probability that in exactly three rounds of tosses, all the coins will have been removed?

{Hint: How can this happen?

3 heads, 3 heads, 3 tails	*3 heads, 2 heads, 2 tails*	*3 heads, 1 head, 1 tail*
2 heads, 2 heads, 2 tails	*2 heads, 1 head, 1 tail*	*1 head, 1 head, 1 tail*

}

In a certain region of the country, a committee is considering an *optimal jury size*. A decision by a jury is made on the basis of a simple majority. If there are an even number of people on a jury, the result could be a tie (hung jury) in which case there is a retrial. Retrials are expensive. Moreover, an incorrect decision on the part of a jury is considered unacceptable. Thus *optimal jury size* to this committee means that size which has the greatest chance of making the correct decision on the first trial when compared to other sizes under consideration.

We'll restrict our considerations to juries of size 2, 3, and 4. Let's suppose there is a probability p, $0 < p < 1$, of an individual on a jury of making a correct decision. Now assuming that these decisions are made on an independent basis, the probability that a two-person will make a correct decision on the first trial is given by p^2. In the case of a three-person jury, a correct decision on the first trial results from one of the following:

Juror 1	**Juror 2**	**Juror 3**	**Probability**
right	right	right	p^3
wrong	right	right	$(1-p)p^2$
right	wrong	right	$p(1-p)p$
right	right	wrong	$p^2(1-p)$

So the probability that a three-person jury will make a correct decision on the first trial is given by $p^3 + 3p^2(1-p)$. In the case of a four-person jury, a correct decision on the first trial results from one of the following:

Juror 1	**Juror 2**	**Juror 3**	**Juror 4**	**Probability**
right	right	right	right	p^4
wrong	right	right	right	$(1-p)p^3$
right	wrong	right	right	$p(1-p)p^2$
right	right	wrong	right	$p^2(1-p)p$
right	right	right	wrong	$p^3(1-p)$

So the probability that a four-person jury will make the correct decision on the first trial is given by $p^4 + 4p^3(1-p)$. The information gained so far is summarized in the following table:

Size of Jury	**Probability of correct decision on first trial**
2	p^2
3	$p^3 + 3p^2(1-p)$
4	$p^4 + 4p^3(1-p)$

Now under the definition of *optimal jury size*, we would choose a two-person jury over a three-person jury if

$$p^2 > p^3 + 3p^2(1-p)$$

Since $p^2 > 0$, we can divide both sides by p^2 to get

$$1 > p + 3(1-p)$$
$$1 > -2p + 3$$
$$2p > 2$$
$$p > 1$$

Since this last inequality is never satisfied, it follows that we would never choose a two-person jury over a three-person jury.

We would choose a four-person jury over a three-person jury if

$$p^4 + 4p^3(1-p) > p^3 + 3p^2(1-p).$$

Again, dividing both sides by p^2, we get

$$p^2 + 4p(1-p) > p + 3(1-p)$$
$$-3p^2 + 6p - 3 > 0$$

Dividing both sides by -3, we get

$$p^2 - 2p + 1 < 0$$
$$(p-1)^2 < 0$$

Since this last inequality is impossible, it follows that we would never pick a four-person jury over a three-person jury.

The previous analysis indicates that a three-person jury is optimal if one considers two, three, and four-person juries.

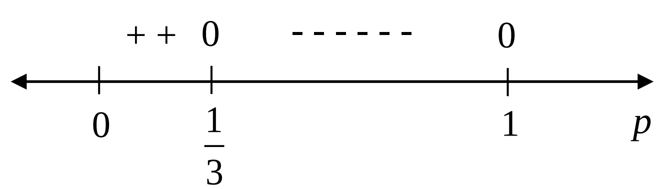
Let's conclude the discussion by comparing a two-person jury to a four-person jury. We would choose a two-person jury over a four-person jury if

$$p^2 > p^4 + 4p^3(1-p).$$

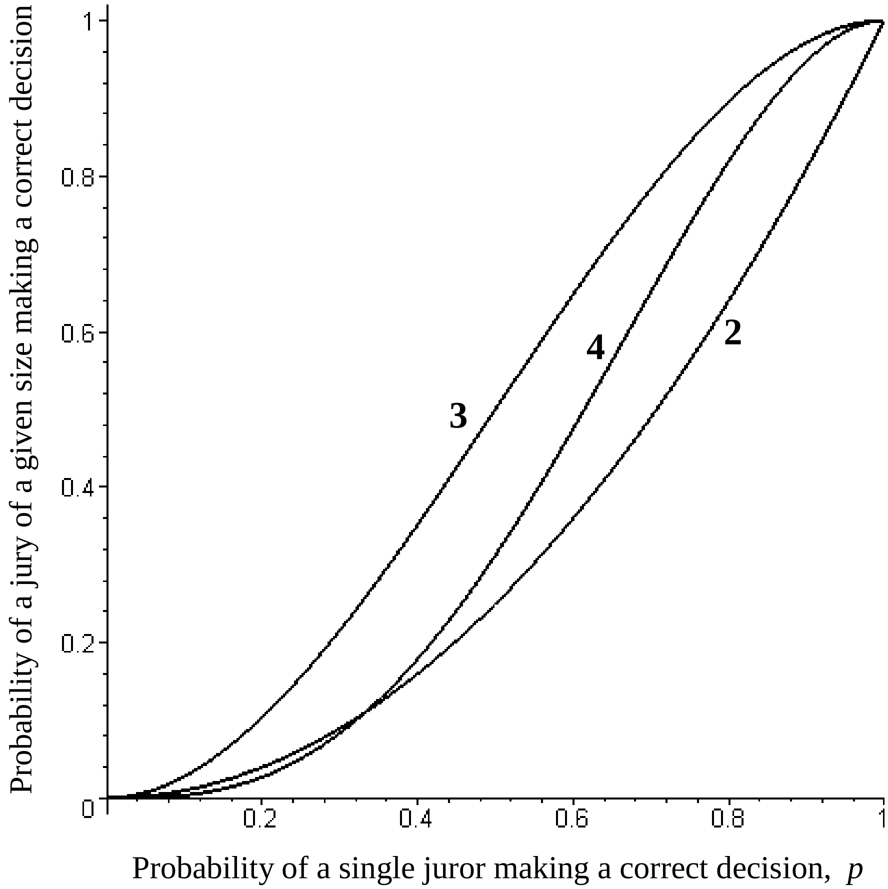
Again, dividing both sides by p^2, we get

$$1 > p^2 + 4p(1-p)$$
$$3p^2 - 4p + 1 > 0$$
$$(3p-1)(p-1) > 0$$

Here's the sign chart for the left side of the last inequality:



We can conclude that we would choose a two-person jury over a four-person jury if $p < \frac{1}{3}$, and a four-person jury over a two-person jury if $p > \frac{1}{3}$. If $p = \frac{1}{3}$, then both have the same chance of making the correct decision on the first trial. If $p > \frac{1}{3}$, and in an educated society one would hope that would happen, a four-person jury would be preferable to a two-person jury. Here is a graph of the probability of a correct decision on the first trial by juries of size two, three, and four.



Trial By Jury.

55. a) Using the definition of *optimal jury size*, which is preferable, a one-person jury or a three-person jury?

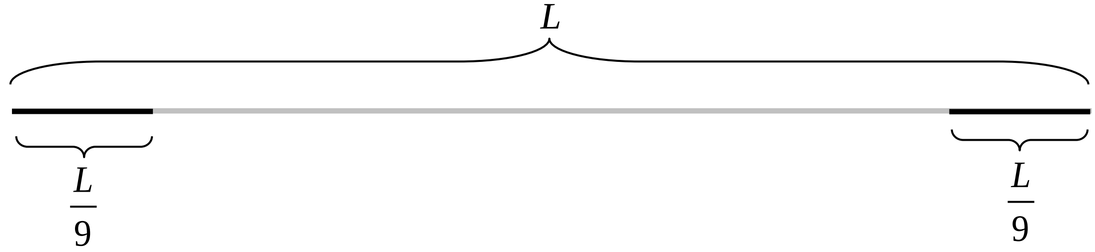
b) Using the definition of *optimal jury size*, which is preferable, a five-person jury or a three-person jury?

c) Using the definition of *optimal jury size*, find the optimal jury size for juries up to size seven.

Just The Rope, Not The Cheese.

56. a) A magician cuts a rope into two pieces at a point selected at random. What is the probability that the length of the longer piece of rope is at least 8 times the length of the shorter piece of rope?

{Hint: If we call the length of the rope L, then the cut would have to be made in the black portion of the rope indicated in the diagram.



}

b) What is the probability that the length of the longer piece of rope is at least 6 times the length of the shorter piece of rope?

Think Of A Letter From A To Z.

57. Each person in a group of people selects a letter of the alphabet at random. What is the fewest number of people that could be in the group if the probability of two or more people choosing the same letter is greater than $\frac{1}{2}$?

{Hint: For the number of people, n, with $n \le 26$, the number of ways that everyone chooses a different letter is $26 \cdot 25 \cdot 24 \cdot \cdots \cdot (26 - n + 1)$ or ${}_{26}P_n$. The number of ways that the n people can choose a letter without restriction is 26^n. So the probability that everyone chooses a different letter is $\frac{{}_{26}P_n}{26^n}$. Now use the complement.}

Consider the problem of estimating the number of fish in a certain lake. One method is the following: Randomly select a spot on the lake and using a net, get a catch of fish. Suppose that 200 fish are caught. These fish are then tagged and returned to the lake. One week later, another spot is selected and 100 fish are netted with 40 of them having tags. This suggests that $\frac{40}{100}$ or .4 of the total number of fish in the lake are tagged. If F is the total number of fish in the lake, then

$$.4F = 200$$

$$F = \frac{200}{.4} \quad .$$

$$F = 500$$

We can generalize the process as follows: Let

$$
\begin{aligned}
N_1 &= \text{the number of fish caught and tagged} \\
N_2 &= \text{the number of fish in the second catch} \\
T &= \text{the number of tagged fish in the second catch} \\
F &= \text{the total number of fish in the lake}
\end{aligned}
$$

Then, as before:

$$\frac{T}{N_2} F = N_1$$

$$F = \frac{N_1 N_2}{T}$$

$\frac{N_1 N_2}{T}$ is called a basic estimate for the total number of fish in the lake, F.

For example, if

$$
\begin{aligned}
N_1 &= \text{the number of fish caught and tagged} = 60 \\
N_2 &= \text{the number of fish in the second catch} = 80 \\
T &= \text{the number of tagged fish in the second catch} = 10
\end{aligned}
$$

Then a basic estimate for the total number of fish in the lake is given by

$$F = \frac{N_1 N_2}{T} = \frac{60 \cdot 80}{10} = 480 .$$

Now let's look at this estimation problem from a different point of view. Suppose we play the same game with a small goldfish pond. Suppose that

$$
\begin{aligned}
N_1 &= \text{the number of fish caught and tagged} = 3 \\
N_2 &= \text{the number of fish in the second catch} = 3 \\
T &= \text{the number of tagged fish in the second catch} = 1
\end{aligned}
\quad .
$$

Our basic estimate for the number of goldfish in the pond is $F = \frac{N_1 N_2}{T} = \frac{3 \cdot 3}{1} = 9$. In general, the smallest number of fish that could be in the lake or pond is $N_1 + N_2 - T$. For this goldfish pond we get $3+3-1=5$. Let's calculate the conditional probability of getting 1 tagged fish our of 3 in the second catch, given that the pond actually contains 5 goldfish.

$$P(\text{1 tagged fish out of 3} | \text{total of 5 fish})$$

$$= \frac{(\text{\# of ways of selecting 1 tagged fish}) \cdot (\text{\# of ways of selecting 2 untagged fish})}{(\text{\# of ways of selecting 3 fish})}$$

$$= \frac{{}_3C_1 \cdot {}_2C_2}{{}_5C_3} = .3.$$

We can also calculate the conditional probability of getting 1 tagged fish out of 3 in the second catch, given that the pond actually contains 6 goldfish.

$$P(\text{1 tagged fish out of 3}|\text{total of 6 fish})$$

$$= \frac{(\text{\# of ways of selecting 1 tagged fish}) \cdot (\text{\# of ways of selecting 2 untagged fish})}{(\text{\# of ways of selecting 3 fish})}$$

$$= \frac{{}_3C_1 \cdot {}_3C_2}{{}_6C_3} = .45.$$

It is not unreasonable to say that it is more likely that the pond contains 6 goldfish rather than 5. In general, the formula for the conditional probability of getting 1 tagged fish out of 3 in the second catch, given that the pond actually contains K goldfish is given by

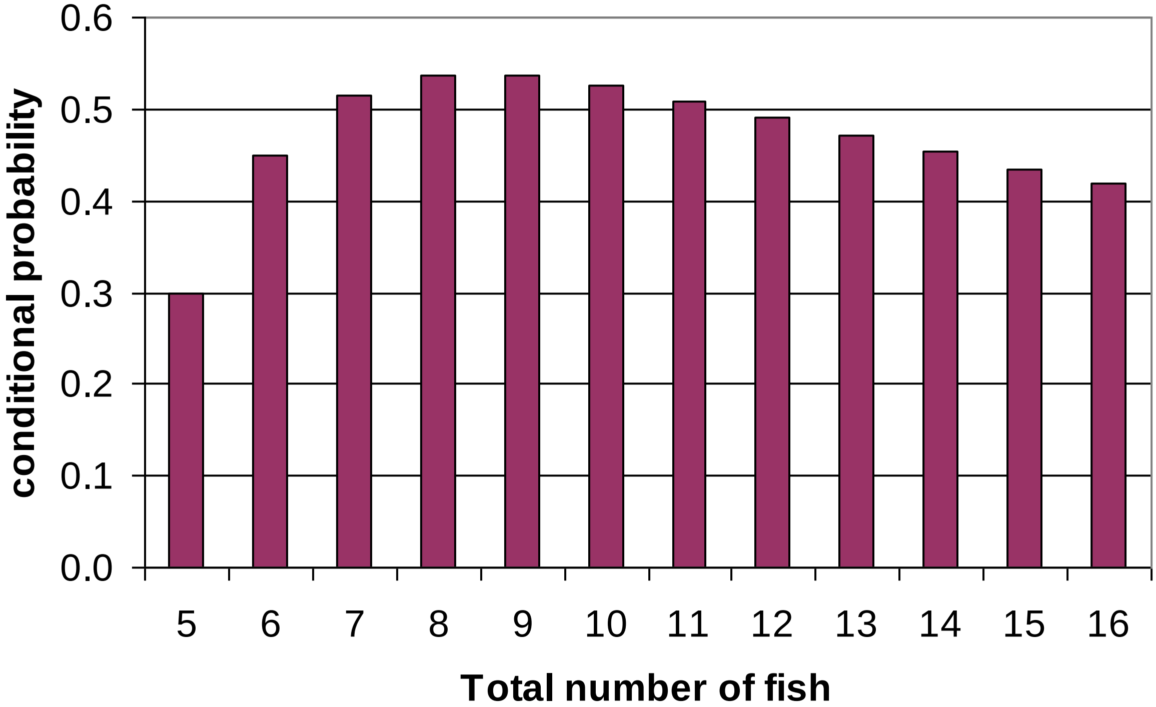
$$P(\text{1 tagged fish out of 3}|\text{total of K fish})$$

$$= \frac{(\text{\# of ways of selecting 1 tagged fish}) \cdot (\text{\# of ways of selecting 2 untagged fish})}{(\text{\# of ways of selecting 3 fish})}$$

$$= \frac{{}_3C_1 \cdot {}_{K-3}C_2}{{}_KC_3}.$$

Here is a table and graph of conditional probabilities for different values of K:

K	$\frac{{}_3C_1 \cdot {}_{K-3}C_2}{{}_KC_3}$
5	0.300000000000
6	0.450000000000
7	0.514285714286
8	0.535714285714
9	0.535714285714
10	0.525000000000
11	0.509090909091
12	0.490909090909
13	0.472027972028
14	0.453296703297
15	0.435164835165
16	0.417857142857



From the previous table and graph, it would appear that the most likely number of goldfish in the pond is 8 or 9. Before accepting this, however, there is one question to answer. If the table were continued, is it possible that at some point the probabilities would begin to increase again? In order to answer this question, let's look at

$$\frac{P(1 \text{ tagged fish out of } 3 | \text{total of K fish})}{P(1 \text{ tagged fish out of } 3 | \text{total of K+1 fish})}$$

which we'll abbreviate as

$$\frac{P(K)}{P(K+1)}.$$

The probabilities in the table will be increasing when $\frac{P(K)}{P(K+1)} < 1$, since $P(K) < P(K+1)$, and decreasing when $\frac{P(K)}{P(K+1)} > 1$, since $P(K) > P(K+1)$. Now we know that $P(K) = \frac{{}_3C_1 \cdot {}_{K-3}C_2}{{}_KC_3}$ and $P(K+1) = \frac{{}_3C_1 \cdot {}_{K-2}C_2}{{}_{K+1}C_3}$, so $\frac{P(K)}{P(K+1)} = \frac{\frac{{}_3C_1 \cdot {}_{K-3}C_2}{{}_KC_3}}{\frac{{}_3C_1 \cdot {}_{K-2}C_2}{{}_{K+1}C_3}} = \frac{K^2 - 3K - 4}{K^2 - 4k + 4}$. If we do a polynomial division, we get that $\frac{P(K)}{P(K+1)} = 1 + \frac{K-8}{K^2 - 4K + 4} = 1 + \frac{K-8}{(K-2)^2}$. So from this we can conclude that $\frac{P(K)}{P(K+1)} < 1$ for $5 \le K < 8$ and $\frac{P(K)}{P(K+1)} > 1$ for $K > 8$. From this we have that the conditional probabilities will continue to decrease beyond 9. Therefore we can say that the most likely number of fish in the pond is 8 or 9. We call 8 or 9 a ***Maximum Likelihood Estimate***. Recall that our basic estimate was 9.

Go Jump In The Pond!

58. Suppose that

$$N_1 = \text{the number of fish caught and tagged} = 3$$
$$N_2 = \text{the number of fish in the second catch} = 4$$
$$T = \text{the number of tagged fish in the second catch} = 2$$

a) What is a basic estimate for the number of fish in the pond?

b) What is the fewest number of fish possible in the pond?

c) What is the most likely number of fish in the pond?

d) Verify that the conditional probabilities will continue to decrease after a certain value.

e) How do the basic estimate and the most likely estimate compare?

Can You Cut It As An Escort?

59. In 1988, the ignition keys for Ford Escorts were made of a blank key with five cuts, each cut made to one of five different depths.

a) How many different key types were there?

b) If in 1988, Ford sold 381,250 Escorts, and assuming that the key types were assigned uniformly to the Escorts, what is the probability that one Escort key will start another one selected at random?

{Hint: If the key types were assigned uniformly, then there would be 122 of each type. So for another car's key to match a given car's type, it would have to be one of the remaining 121 keys of that type.}

c) If an Escort is selected, what is the probability that a randomly selected key type will start it?

The answers in b) and c) aren't close by coincidence. Assuming a uniform assignment of the key types, the probability that a randomly selected Escort key will start another one is $\frac{\frac{N}{3{,}125}-1}{N-1}$, where N is the total number of Escorts sold. Notice that $\frac{\frac{N}{3{,}125}-1}{N-1}=\frac{\frac{1}{3{,}125}-\frac{1}{N}}{1-\frac{1}{N}}$, so for large values of N, $\frac{1}{N}\approx 0$, and $\frac{\frac{N}{3{,}125}-1}{N-1}=\frac{\frac{1}{3{,}125}-\frac{1}{N}}{1-\frac{1}{N}}\approx\frac{\frac{1}{3{,}125}}{1}=\frac{1}{3{,}125}$.

Unrandomly Choose A Random 3-digit Number.

60. Sometimes things which seem random really aren't. For example, a magician writes a number and seals it in an envelope. He asks you to think of a 3-digit number with all the digits the same, but you don't tell him the number. Then he asks you to add the three digits together and divide the original number by this sum. The magician opens the envelope, and shows you the result that you got. What number did the magician seal in the envelope? Show that it will always work.

Hint: If the number is aaa, then its value is $100a+10a+a=111a$. *The sum of the digits is* $a+a+a=3a$. *If you divide the number by the sum of the digits, you get* $\frac{111a}{3a}$.

Being Unfair With Fair Dice.

61. Sometimes things which seem random really aren't. For example, a magician writes a number and seals it in an envelope. He asks you to roll two standard dice. Then he asks you to multiply the top two numbers, multiply the bottom two numbers, multiply the top of one by the bottom of the other, and multiply the other top and bottom. The magician then asks you to add up the four answers. When the magician opens the envelope, it's the same number. What number did the magician seal in the envelope? Show that it will always work.

Hint: Suppose that the numbers are t_1, b_1, t_2, b_2. *On a standard die, opposite numbers add to 7, so* $t_1 + b_1 = 7$ *and* $t_2 + b_2 = 7$. *The sum of the four products is* $t_1t_2 + b_1b_2 + t_1b_2 + t_2b_1$. *Notice that* $t_1t_2 + b_1b_2 + t_1b_2 + t_2b_1 = (t_1t_2 + t_1b_2) + (t_2b_1 + b_1b_2) = t_1(t_2 + b_2) + b_1(t_2 + b_2)$.

I Won't Just Give You One; I'll Give You Both.

62. Sometimes things which seem random really aren't. For example, have a friend secretly choose a number from 1 to 9 and another number from 10 to 99. Have your friend add the two numbers. Multiply the result by 5. Add the smaller chosen number. Multiply this sum by 2. Subtract the smaller chosen number. Have your friend give you the result. You should be able to determine the two numbers. Describe how to determine the two numbers. Show that it will always work.

Hint: Suppose that the first number is x and the second number is y. Then here's what happens:

$2[5(x+y)+x]-x$. *Notice that* $2[5(x+y)+x]-x = 11x + 10y = 10\cdot\underbrace{(x+y)}_{\text{remaining digits to the left}} + \underbrace{x}_{\text{one's digit}}$.

Given a set of numbers $\{a_1, a_2, a_3, \ldots, a_n\}$, we would like to find a value, x, which minimizes the sum of its distances from the set of numbers, $|x-a_1| + |x-a_2| + \cdots + |x-a_n|$, and also a value x, which minimizes the sum the squares of its distances from the set of numbers, $(x-a_1)^2 + (x-a_2)^2 + \cdots + (x-a_n)^2$.

Let's start with minimizing the sum of the squares of the distances:

$$(x-a_1)^2 + (x-a_2)^2 + \cdots + (x-a_n)^2 = (x^2 - 2a_1x + a_1^2) + (x^2 - 2a_2x + a_2^2) + \cdots + (x^2 - 2a_nx + a_n^2)$$
$$= nx^2 - 2(a_1 + a_2 + \cdots + a_n)x + (a_1^2 + a_2^2 + \cdots + a_n^2)$$

Now let's complete the square on the right side:

Factor out the n from the first two terms.

$$= n\left[x^2 - \frac{2(a_1 + a_2 + \cdots + a_n)}{n}x\right] + \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)$$

Then add the square of half the coefficient of x inside the brackets, and subtract n times this amount on the outside to balance out this addition.

$$= n\left\{x^2 - \frac{2(a_1 + a_2 + \cdots + a_n)}{n}x + \left[\frac{(a_1 + a_2 + \cdots + a_n)}{n}\right]^2\right\} + \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) - \frac{(a_1 + a_2 + \cdots + a_n)^2}{n}$$

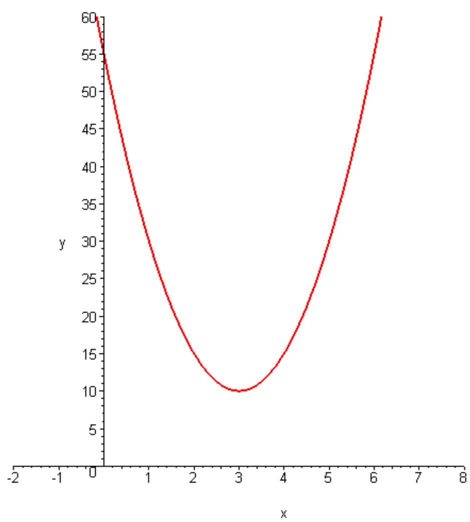
$$= n\left[x - \frac{(a_1 + a_2 + \cdots + a_n)}{n}\right]^2 + \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) - \frac{(a_1 + a_2 + \cdots + a_n)^2}{n}$$

The Lesser Known Of The X-men, The Square Minimizer.

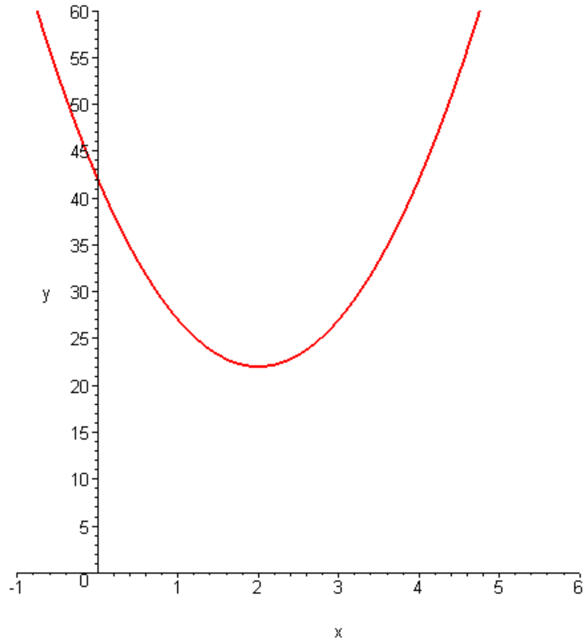
63. So what is the special value of x that will minimize the right side of the equation, and what is this minimum value?

Let's look at the graph of the sum of the squared distances for some particular sets of numbers:

For $\{1,2,3,4,5\}$, the graph of $(x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + (x-5)^2$ is the following.

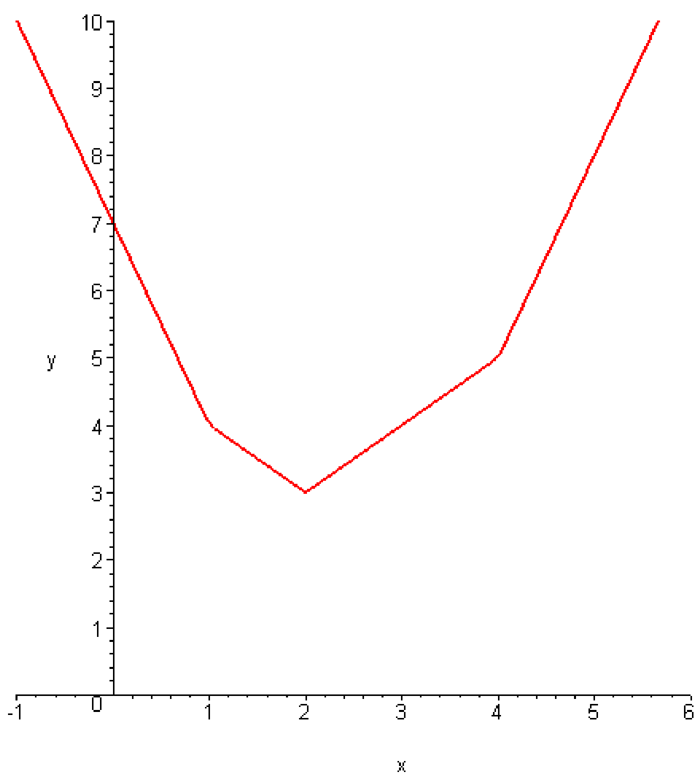


For $\{-1,0,3,4,4\}$, the graph of $(x+1)^2 + x^2 + (x-3)^2 + (x-4)^2 + (x-4)^2$ is the following.



Now let's minimize the sum of the distances, $|x-a_1|+|x-a_2|+\cdots+|x-a_n|$. In this case, we'll look at some specific examples, and we'll look for a pattern.

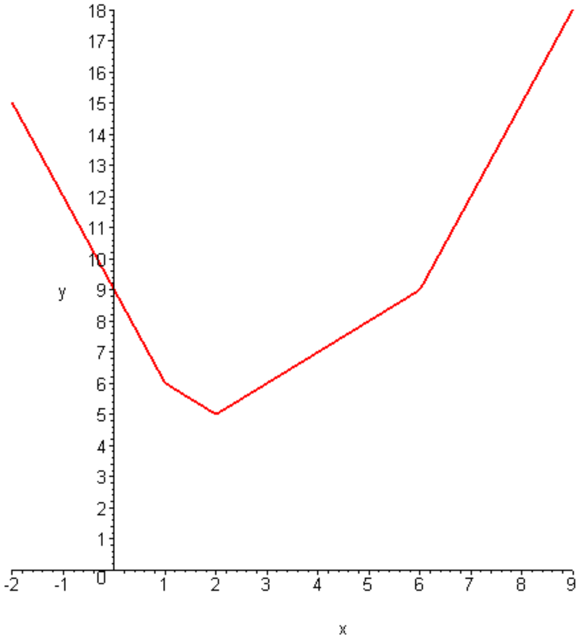
For $\{1,2,4\}$, the graph of $|x-1|+|x-2|+|x-4|$ is the following.



The piecewise formula is

$$|x-1|+|x-2|+|x-4|=\begin{cases}-3x+7; x\le 1\\ -x+5\ ; 1\le x\le 2\\ x+1\ ; 2\le x\le 4\\ 3x-7\ ; x\ge 4\end{cases}$$

For $\{1,2,6\}$, the graph of $|x-1|+|x-2|+|x-6|$ is the following.

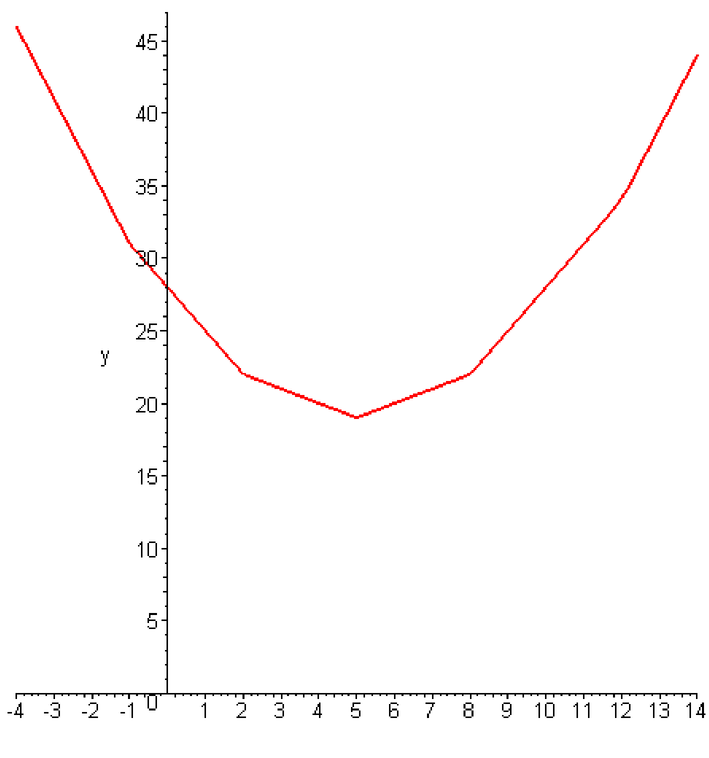


The piecewise formula is

$$|x-1|+|x-2|+|x-6|=\begin{cases}-3x+9; x\le 1\\ -x+7\ ;1\le x\le 2\\ x+3\ ;2\le x\le 6\\ 3x-9\ ;x\ge 6\end{cases}$$

For $\{-1,2,5,8,12\}$, the graph of $|x+1|+|x-2|+|x-5|+|x-8|+|x-12|$ is the following.

The piecewise formula is $$|x+1|+|x-2|+|x-5|+|x-8|+|x-12|=\begin{cases}-5x+26; x\le -1\\ -3x+28\ ;-1\le x\le 2\\ -x+24\ ;2\le x\le 5\\ x+14\ ;5\le x\le 8\\ 3x-2\ ;8\le x\le 12\\ 5x-26\ ;x\ge 12\end{cases}$$



In these three examples each set contained an odd number of values.

Another Lesser Known Of The X-men, The Odd Absolute Minimizer.

64. What is the value of *x* that minimizes the sum of the distances in the case of an odd number of values?

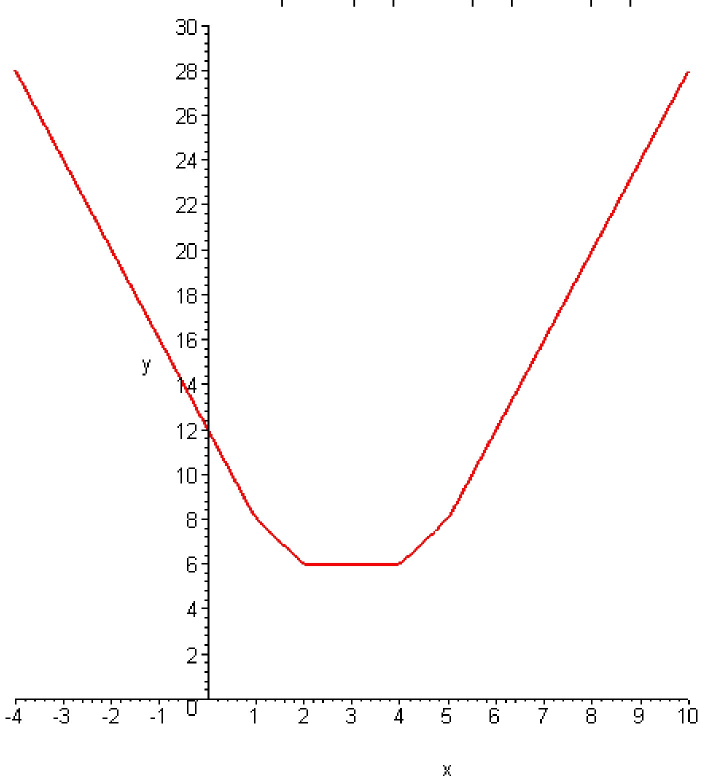
Determine minimizing values of *x* for the sums of the distances for the following sets:

a) $\{1,2,3,4,5,6,7\}$ **b)** $\{-1,2,3,-4,5,-6,7\}$

Let's do an example with an even number of values:

For $\{1,2,4,5\}$, the graph of $|x-1|+|x-2|+|x-4|+|x-5|$ is the following.

The piecewise formula is $|x-1|+|x-2|+|x-4|+|x-5| = \begin{cases} -4x+12 & ;x \le 1 \\ -2x+10 & ;1 \le x \le 2 \\ 6 & ;2 \le x \le 4 \\ 2x-2 & ;4 \le x \le 5 \\ 4x-12 & ;x \ge 5 \end{cases}$



Another Lesser Known Of The X-men, The Even Absolute Minimizer.

65. What's the value of *x* that minimizes the sum of the distances if there are an even number of values?

Determine minimizing values of *x* for the sums of the distances for the following sets:

a) $\{1, 2, 3, 4, 5, 6\}$ **b)** $\{1, 2, 3, 3, 5, 6\}$

The Electric Company On Sesame Street.

66. An electrical station is to be located on a straight street containing six houses. House A is located on one end of the street, House B is located 500 feet from House A, House C is located 600 feet from House A, House D is located 800 feet from House A, House E is located 950 feet from House A, and House F is located 1200 feet from House A. How far from House A on the street should the station be located to use the least amount of wire connecting it to each of the 6 houses?

Seven matchboxes are arranged in a row, as in the following diagram. The first contains 19 matches, the second 9, and the remaining ones contain 26, 8, 18, 11, and 14 matches, respectively. Matches may be taken from any matchbox and transferred to any adjacent matchbox. The matches must be redistributed so that the number of matches in each matchbox is the same. How can this be accomplished, shifting as few matches as possible?

Matchbox 1	19
Matchbox 2	9
Matchbox 3	26
Matchbox 4	8
Matchbox 5	18
Matchbox 6	11
Matchbox 7	14

There is a total of 105 matches in all the boxes. So in order to have the same number of matches in each of the seven matchboxes, each box must contain 15 matches. The solution of the problem is to shift 4 matches from box 1 to box 2.

Matchbox 1	15
Matchbox 2	13
Matchbox 3	26
Matchbox 4	8
Matchbox 5	18
Matchbox 6	11
Matchbox 7	14

Then shift 2 matches from box 3 to box 2 and 9 matches from box 3 to box 4.

Matchbox 1	15
Matchbox 2	15
Matchbox 3	15
Matchbox 4	17
Matchbox 5	18
Matchbox 6	11
Matchbox 7	14

Then shift 2 matches from box 4 to box 5 and 5 matches from box 5 to box 6.

Matchbox 1	15
Matchbox 2	15
Matchbox 3	15
Matchbox 4	15
Matchbox 5	15
Matchbox 6	16
Matchbox 7	14

Finally, shift 1 match from box 6 to box 7.

Matchbox 1	15
Matchbox 2	15
Matchbox 3	15
Matchbox 4	15
Matchbox 5	15
Matchbox 6	15
Matchbox 7	15

The total number of shifted matches is $4+2+9+2+5+1=23$.

An algebraic approach that leads to this result is the following: If matches are shifted from a smaller numbered box to a larger numbered box, then it's considered a positive shift, otherwise, it's considered a negative shift. Shift x matches from box 1 to box 2. This gives $19-x$ matches in box 1 and $9+x$ matches in box 2. In order to have 15 matches in box 2, we need to shift $x-6$ matches from box 2 to box 3. This gives 15 matches in box 2 and $x+20$ matches in box 3. In order to have 15 matches in box 3, we need to shift $x+5$ matches from box 3 to box 4. This gives 15 matches in box 3 and $x+13$ matches in box 4. In order to have 15 matches in box 4, we need to shift $x-2$ matches from box 4 to box 5. This gives 15 matches in box 4 and $x+16$ matches in box 5. In order to have 15 matches in box 5, we need to shift $x+1$ matches from box 5 to box 6. This gives 15 matches in box 5 and $x+12$ matches in box 6. In order to have 15 matches in box 6, we need to shift $x-3$ matches from box 6 to box 7. This gives 15 matches in box 6 and $x+11$ matches in box 7. So the results are the following:

Matchbox 1	$19-x$	x
Matchbox 2	15	$x-6$
Matchbox 3	15	$x+5$
Matchbox 4	15	$x-2$
Matchbox 5	15	$x+1$
Matchbox 6	15	$x-3$
Matchbox 7	$x+11$	

The total number of shifted matches is $|x|+|x-6|+|x+5|+|x-2|+|x+1|+|x-3|$, and in order to have 15 matches in each box, we need $19-x=15$ and $x+11=15$. This leads to $x=4$ and the number of shifted matches being $|4|+|4-6|+|4+5|+|4-2|+|4+1|+|4-3|=23$.

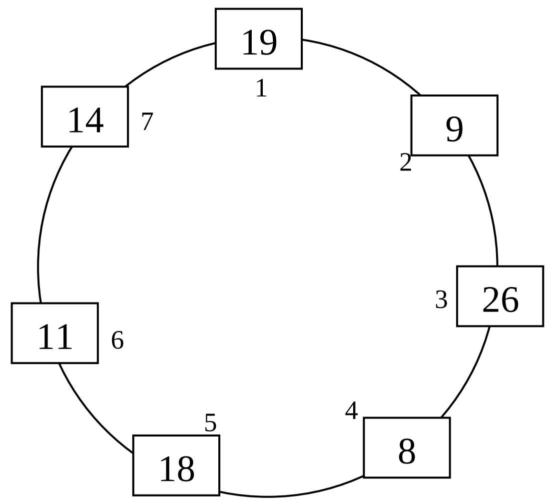
Match Game 2020.

67. Consider the following arrangement of matchboxes along with the given number of matches in each box:

Matchbox 1	1
Matchbox 2	2
Matchbox 3	3
Matchbox 4	72
Matchbox 5	32
Matchbox 6	20
Matchbox 7	10

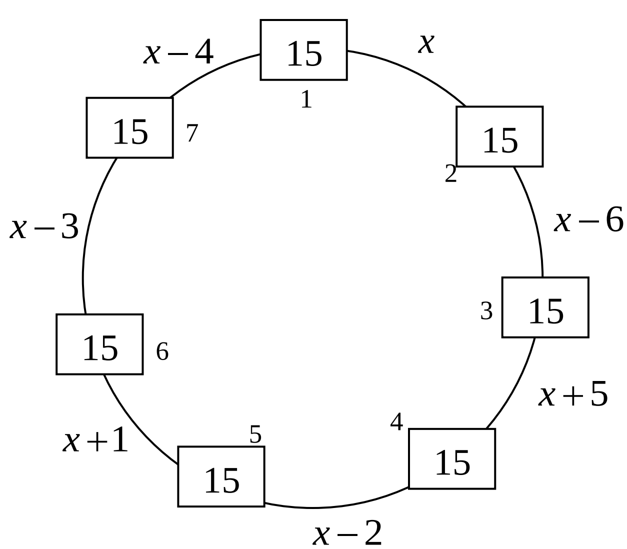
Matches may be taken from any matchbox and transferred to any adjacent matchbox. The matches must be redistributed so that the number of matches in each matchbox is the same. How can this be accomplished, shifting as few matches as possible?

Now suppose that the seven matchboxes are arranged in a circle:



It is permissible to move matches from any box to any adjacent box. The matches must be shifted in such a way that the number of matches in all the boxes becomes the same. How can this be done, shifting as few matches as possible?

An algebraic approach is the following: If matches are shifted from a smaller numbered box to a larger numbered box, then it's considered a positive shift, otherwise, it's considered a negative shift. Shift x matches from box 1 to box 2. This gives $19-x$ matches in box 1 and $9+x$ matches in box 2. In order to have 15 matches in box 2, we need to shift $x-6$ matches from box 2 to box 3. This gives 15 matches in box 2 and $x+20$ matches in box 3. In order to have 15 matches in box 3, we need to shift $x+5$ matches from box 3 to box 4. This gives 15 matches in box 3 and $x+13$ matches in box 4. In order to have 15 matches in box 4, we need to shift $x-2$ matches from box 4 to box 5. This gives 15 matches in box 4 and $x+16$ matches in box 5. In order to have 15 matches in box 5, we need to shift $x+1$ matches from box 5 to box 6. This gives 15 matches in box 5 and $x+12$ matches in box 6. In order to have 15 matches in box 6, we need to shift $x-3$ matches from box 6 to box 7. This gives 15 matches in box 6 and $x+11$ matches in box 7. In order to have 15 matches in box 7, we need to shift $x-4$ matches from box 7 to box 1. This gives 15 matches in box 7 and 15 matches in box 1. So the results are the following:



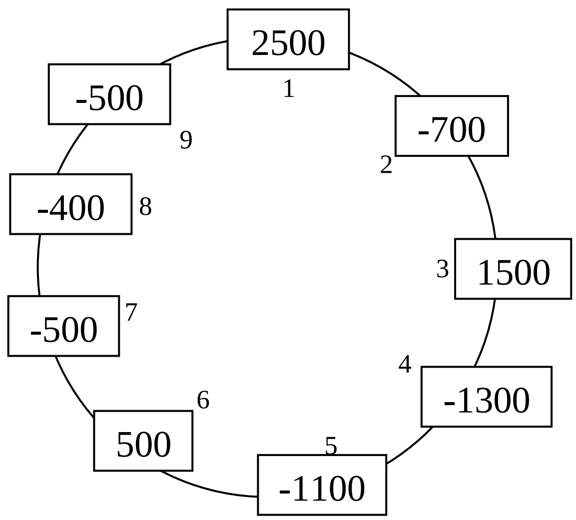
The total number of shifted matches is $|x|+|x-6|+|x+5|+|x-2|+|x+1|+|x-3|+|x-4|$.

Will It Go Round In Circles?

68. Solve the previous problem of the seven matchboxes arranged in a circle.

Here's The Problem You've Been Training For.

69. This problem is similar to the previous problems. Consider a circular railroad with nine evenly spaced stations as in the figure.



At some stations there are coal storages(the positive values), and at the others there are coal users(the negative values). Determine the most economical transportation plan to supply all the users with their needed amounts of coal.

Just Make Something Up.

70. a) Make up a data set of twelve numbers between 0 and 10(inclusive) with these properties:

No number appears more than three times

The mean of the data set is 7

The median of the data set is 7

b) By changing no more than two numbers in the previous data set, make it have a mean of 7, but a median of 8. You may have to redo part a) to make this work.

c) By changing no more than two numbers in the previous data set, make it have a median of 7, but a mean of 6. You may have to redo parts a) and/or b) to make this work.

I Just Want To Raise My Grade.

71. A student received the following quiz grades: 90, 83, 80, 71, 77, 63, 77, and 90.

a) What is the lowest grade the student can receive on the next quiz that would raise the student's mean quiz grade? (Assume that quiz grades are whole numbers from 0 to 100.)

b) What is the highest grade the student can receive on the next quiz that would lower the student's mean quiz grade? (Assume that quiz grades are whole numbers from 0 to 100.)

c) What is the lowest grade the student can receive on the next quiz that would raise the student's median quiz grade? (Assume that quiz grades are whole numbers from 0 to 100.)

d) What is the highest grade the student can receive on the next quiz that would lower the student's median quiz grade? (Assume that quiz grades are whole numbers from 0 to 100.)

The Battle Of The Sexes.

72. In a large math class of 23 men and 25 women, the mean grade on the last test for the women was 89, and for the men was 83. Is it possible to compute the mean test grade for the entire class? If so, do it; if not, explain why.

Beware Of Underreporting.

73. A survey of 100 recent college graduates was made to determine the mean salary of recent college graduates. The mean salary was found to be $35,000. It turns out that one of the graduates incorrectly answered the survey. He said he is earning $29,000 when, in fact, he is earning $42,000. What is the actual mean salary of the 100 graduates?

How Can Square-rooting Make It Bigger?

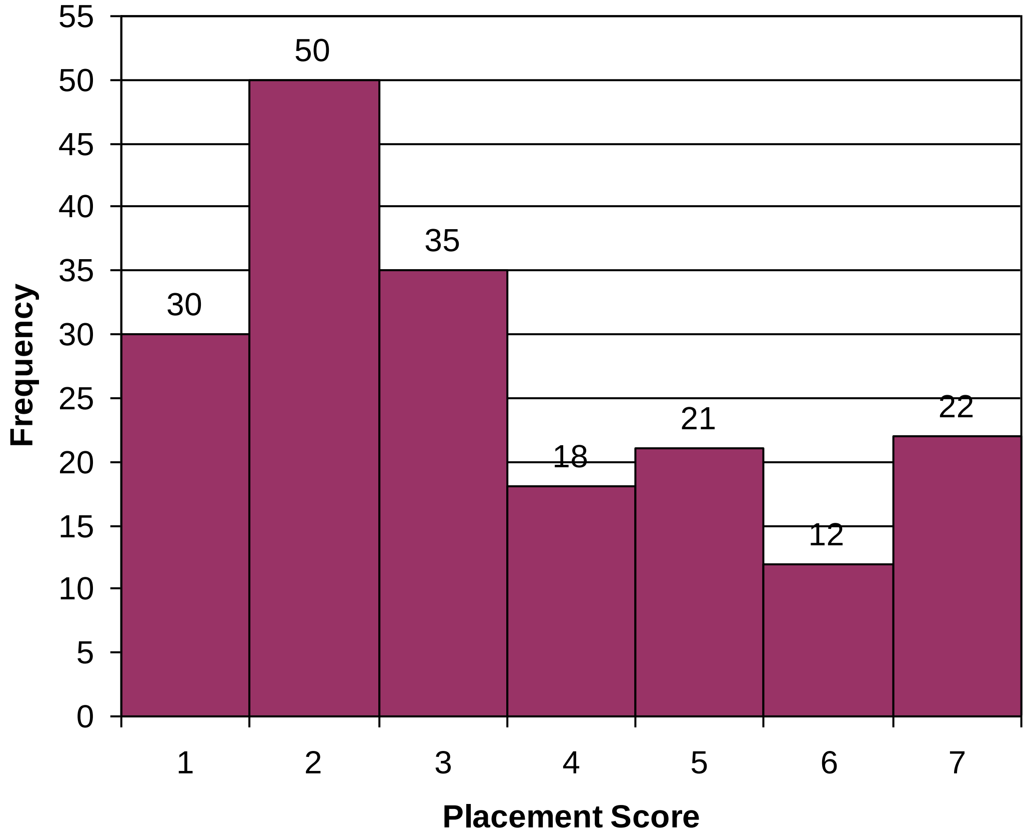
74. Is it possible for the variance to be less than the standard deviation for a set of data? If it is possible, create a data set of three numbers so that the variance is less than the standard deviation.

How Can Square-rooting Make It The Same?

75. Is it possible for the variance to equal the standard deviation for a data set? If it is possible, create a data set of three <u>*different*</u> numbers so that the variance is equal to the standard deviation.

Mean And Median Ala Mode.

76. The following histogram shows the scores on a placement test. Which of the following statements applies to the scores? Explain.



a) The mode and the mean are the same.
b) The mode is greater than the mean.
c) The mode and the median are the same.
d) The mode is less than the median.

Just Give Him The Slip.

77. a) Suppose you have a bag with 12 slips of paper in it. Some of the slips have a 2 on them, and the rest have a 7. If the expected value of the number shown on a randomly drawn slip is 3.25, then how many of the slips have a 2?

{Hint:

x	2	7
$p(x)$	$\frac{n}{12}$	$\frac{12-n}{12}$

}

b) If the expected value of the number shown on a randomly drawn slip is 4.5, then how many of the slips have a 2?

Risky Business.

78. a) You wish to invest \$1,000, and you have two choices. One is a sure thing: You will make a 5% profit. The other is a riskier venture. If the riskier venture pays off, you will make a 25% profit; otherwise, you lose your \$1,000. What is the minimum required probability of the riskier venture paying off in order for its expected value to equal or exceed the value of the first investment?

{Hint:

sure thing

x	\$1,050	-\$1,000
$p(x)$	1	0

risky venture

x	\$1,250	-\$1,000
$p(x)$	p	$1-p$

}

b) If the riskier venture pays off, you will make a 20% profit; otherwise, you lose your \$1,000. What is the minimum required probability of the riskier venture paying off in order for its expected value to equal or exceed the value of the first investment?

It's Kinda Fishy.

79. a) Twenty-four tagged fish are added to a pond containing fish. Later, 10 random samples of ten fish from the pond are collected with replacement after each fish is caught, and the average number of tagged fish per 10 is 2. Estimate the original fish population of the pond using expected value.

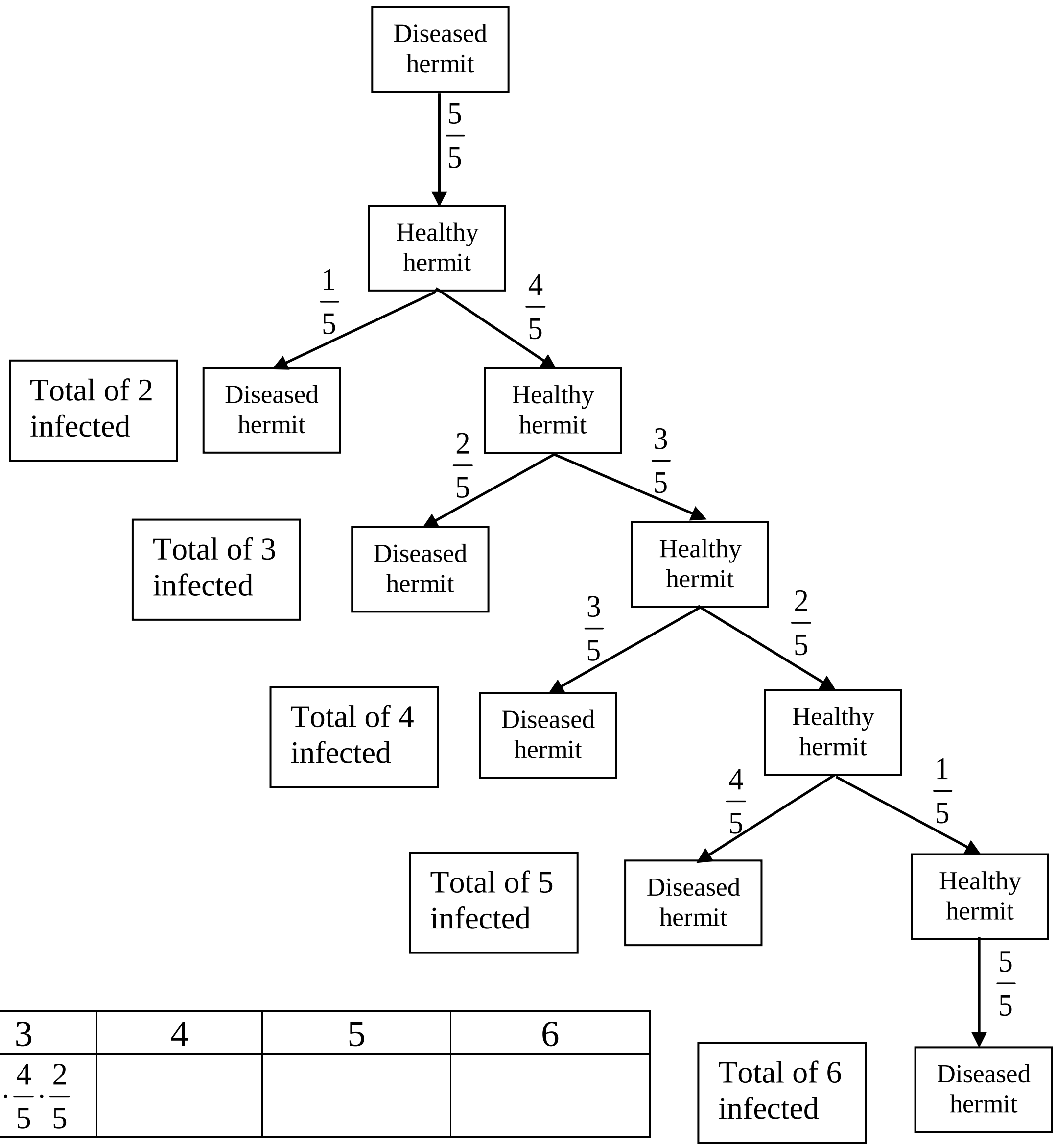
{Hint: Let N be the original number of fish in the pond. After adding 24 tagged fish, the probability of selecting a tagged fish is $\frac{24}{N+24}$. *The expected number of tagged fish caught from 10 tries is* $10\cdot\frac{24}{N+24}=\frac{240}{N+24}$, *and 2 tagged fish were caught on average per 10 fish caught.}*

b) Later, 10 random samples of ten fish from the pond are collected with replacement after each fish is caught, and the average number of tagged fish per 10 is 1.5. Estimate the original fish population of the pond using expected value.

Germin' Hermits.

80. Six (unusually sociable) hermits live on an otherwise deserted island. An infectious disease strikes the island. The disease has a 1-day infectious period and after that the person is immune (cannot get the disease again). Assume one of the hermits gets the disease. He randomly visits one of the other hermits during his infectious period. If the visited hermit has not had the disease, he gets it and is infectious the following day. The visited hermit then visits another hermit. The disease is transmitted until an infectious hermit visits an immune hermit, and the disease dies out. There is one hermit visit per day. Assuming this pattern of behavior, how many hermits can be expected, on the average, to get the disease?

{Hint:



X	2	3	4	5	6
$p(X)$	$\frac{5}{5}\cdot\frac{1}{5}$	$\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{2}{5}$			

}

Unexpected Expectations.

81. For the following probability distribution,

X	1	5	8	11	15
$p(X)$					

a) Assign probabilities so that the expected value of X is 5 and ***probabilities of 0 are not permitted.***

{See the Unexpected Expectations link.}

b) Assign probabilities so that the expected value of X is 8 and ***probabilities of 0 are not permitted.***

c) Assign probabilities so that the expected value of X is 11 and ***probabilities of 0 are not permitted.***

d) Assign probabilities so that the expected value of X is 7 and ***probabilities of 0 are not permitted.***

e) Assign probabilities so that the expected value of X is equal 1.

f) Assign probabilities so the expected value of X is equal to 15.

g) Can you assign probabilities so that the expected value of X is 0? Explain.
{Hint: Suppose that

X	1	5	8	11	15
$p(X)$	a	b	c	d	e

With $a,b,c,d,e \geq 0$ *and* $a+b+c+d+e=1$

$E(X)=a+5b+8c+11d+15e \geq a+b+c+d+e \geq ?.\}$

h) Can you assign probabilities so that the expected value of X is 16? Explain.
{Hint: Suppose that

X	1	5	8	11	15
$p(X)$	a	b	c	d	e

With $a,b,c,d,e \geq 0$ *and* $a+b+c+d+e=1$

$E(X)=a+5b+8c+11d+15e \leq 15a+15b+15c+15d+15e \leq ?.\}$

See the Unexpected Expectations Link!

Expected Value of a sum of random variables:

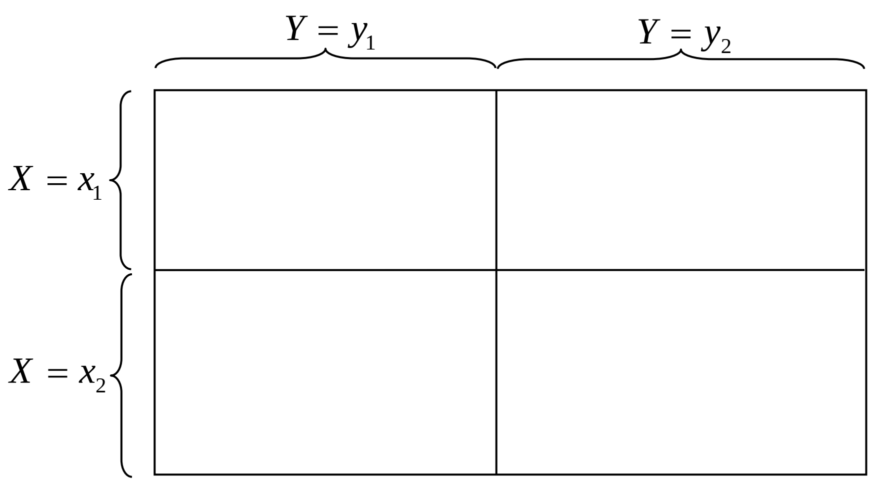
If X and Y are random variables, X takes on the values $\{x_1, x_2\}$, and Y takes on the values $\{y_1, y_2\}$, then

$$\begin{aligned} E(X+Y) &= (x_1+y_1)P(X=x_1 \cap Y=y_1) + (x_1+y_2)P(X=x_1 \cap Y=y_2) \\ &\quad + (x_2+y_1)P(X=x_2 \cap Y=y_1) + (x_2+y_2)P(X=x_2 \cap Y=y_2) \end{aligned}$$

. This rearranges into

$$\begin{aligned} E(X+Y) &= x_1\left[P(X=x_1 \cap Y=y_1) + P(X=x_1 \cap Y=y_2)\right] \\ &\quad + x_2\left[P(X=x_2 \cap Y=y_1) + P(X=x_2 \cap Y=y_2)\right] \\ &\quad + y_1\left[P(X=x_1 \cap Y=y_1) + P(X=x_2 \cap Y=y_1)\right] \\ &\quad + y_2\left[P(X=x_1 \cap Y=y_2) + P(X=x_2 \cap Y=y_2)\right] \end{aligned}.$$

But considering the sample space,



It's also equivalent to

$$\begin{aligned} E(X+Y) &= x_1P(X=x_1) + x_2P(X=x_2) + y_1P(Y=y_1) + y_2P(Y=y_2) \\ &= E(X) + E(Y) \end{aligned}.$$

This result holds in general: $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$.

Keep On Rolling The Bones.

82. Calculate the expected sum when 10 rolls of a fair die are made.

{Hint: See the previous discussion.}

Who's Gonna Throw Their Hats Into The Ring?

83. Ten men throw their hats into the center of a room. The hats are mixed up, and each man is randomly given one of the hats. Find the expected number of men who get their own hats. For example, if there are two men, then here are the possible distributions of the hats:

Hat 1	**Hat 2**
Man 1	**Man 2**

Hat 2	**Hat 1**
Man 1	**Man 2**

If X is the number of men who get their own hat, then

X	0	2
$P(X)$	$\frac{1}{2}$	$\frac{1}{2}$

$E(X) = 0 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 1.$

{Hint: Let $X_1 = \begin{cases} 1; \text{if the first man selects his hat} \\ 0; \text{otherwise} \end{cases}$,

$X_2 = \begin{cases} 1; \text{if the second man selects his hat} \\ 0; \text{otherwise} \end{cases}, \ldots, X_{10} = \begin{cases} 1; \text{if the tenth man selects his hat} \\ 0; \text{otherwise} \end{cases}$. *Then the expected number of men that select their own hats is* $E(X_1 + X_2 + \cdots + X_{10})$*, so use the previous discussion.}*

Lotsa Lottery.

84. Let *X* represent your winnings from one ticket of lottery #1 and *Y* represent your winnings from one ticket of lottery #2.

X	\$100	\$10	-\$1
P(X)	.01	.1	.89

Y	\$200	\$20	5\$	-\$2
P(Y)	.01	.1	.2	.69

Find your expected winnings if you buy 2 tickets for lottery #1 and 3 tickets from lottery #2.

{Hint: See the previous discussion.}

You're Just A Player!

85. a) Two people, A and B, play a game in which a fair coin is repeatedly tossed. If it shows heads, then A wins \$1 from B. If it shows tails, then B wins \$1 from A. The game is played until one of the players runs out of money. If A starts with \$500, and B starts with \$1,000, what is the probability that A will win all of B's money.

Let *X* be the amount of money that A wins at the end of the game. Let X_i be A's winnings on the i^{th} play of the game, $X_i = \begin{cases} 1; \text{coin shows heads} \\ -1; \text{coin shows tails} \end{cases}$. If the game is complete after *n* tosses, then $X = X_1 + X_2 + \cdots + X_n$. From the previous discussion, $E(X) = E(X_1) + E(X_2) + \cdots + E(X_n)$, and $E(X_i) = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$, so $E(X) = 0$.

We also know that

X	1,000	-500
$P(X)$	p	$1-p$

Calculate $E(X)$ the usual way to figure out the probability that A wins all of B's money.

b) If the coin isn't fair and the winnings are different-A wins \$1 from B with probability $\frac{2}{3}$ and B wins \$2 from A with probability $\frac{1}{3}$, then what is the probability that A will win all of B's money?

Conditional Expected Value:

Sometimes it is easier to calculate $E(X)$ *by using conditional expected values. Suppose the random variable Y takes on the values 1, 2, or 3, and the random variable X takes on the values 4 or 5. Then here are the definitions of the conditional expected values of X:*

$$E[X|Y=1]=4\cdot P(X=4|Y=1)+5\cdot P(X=5|Y=1)$$
$$E[X|Y=2]=4\cdot P(X=4|Y=2)+5\cdot P(X=5|Y=2)$$
$$E[X|Y=3]=4\cdot P(X=4|Y=3)+5\cdot P(X=5|Y=3)$$

From these and the definition of conditional probability we can conclude that

$$E[X|Y=1]P(Y=1)=4\cdot P(X=4\cap Y=1)+5\cdot P(X=5\cap Y=1)$$
$$E[X|Y=2]P(Y=2)=4\cdot P(X=4\cap Y=2)+5\cdot P(X=5\cap Y=2)$$
$$E[X|Y=3]P(Y=3)=4\cdot P(X=4\cap Y=3)+5\cdot P(X=5\cap Y=3)$$

So we get that

$$E[X|Y=1]P(Y=1)+E[X|Y=2]P(Y=2)+E[X|Y=3]P(Y=3)$$
$$=4\left[P(X=4\cap Y=1)+P(X=4\cap Y=2)+P(X=4\cap Y=3)\right]$$
$$+5\left[P(X=5\cap Y=1)+P(X=5\cap Y=2)+P(X=5\cap Y=3)\right]$$
$$=4\cdot P(X=4)+5\cdot P(x=5)=E(X)$$

So the conditional expectation formula is
$E(X)=E[X|Y=1]P(Y=1)+E[X|Y=2]P(Y=2)+E[X|Y=3]P(Y=3)$. *You may use this idea to solve the following problems:*

Prison Break.

86. A prisoner is trapped in a dark cell containing three doors. The first door leads to a tunnel that returns him to his cell after 2 days travel. The second to a tunnel that returns him to his cell after 4 days travel. The third door leads to freedom after 1 day of travel. If it is assumed that the prisoner will always select doors 1, 2, and 3 with respective probabilities .5, .3, and .2, what is the expected number of days until the prisoner reaches freedom?

{Hint: Let X be the number of days until the prisoner escapes.

$$E(X)=E[X|\text{door 1}]P(\text{door 1})+E[X|\text{door 2}]P(\text{door 2})+E[X|\text{door 3}]P(\text{door 3})$$

$$=\left[E(X)+2\right]P(\text{door 1})+\left[E(x)+4\right]P(\text{door 2})+1\cdot P(\text{door 3})$$
$$=\left[E(X)+2\right]\cdot(.5)+\left[E(x)+4\right]\cdot(.3)+1\cdot(.2)$$

}

$$=$$

Trapped Like A Rat.

87. A rat is trapped in a maze. Initially, he has to choose one of two directions. If he goes to the right, then he will wander around in the maze for three minutes and will then return to his initial position. If he goes to the left, then with probability 1/3, he will depart the maze after two minutes, and with probability 2/3 he will return to his initial position after five minutes. Assuming that the rat is at all times equally likely to go to the left or right, what is the expected number of minutes that he will be trapped in the maze?

{Hint: Let X be the number of minutes until the rat escapes.

$$E(X) = E[X \mid right]P(\text{right}) + E[X \mid \text{bad left}]P(\text{bad left}) + E[X \mid \text{good left}]P(\text{good left})$$

$$= [E(X)+3]P(\text{right}) + [E(x)+5]P(\text{bad left}) + 2\cdot P(\text{good left})$$

$$= [E(X)+3]\cdot\frac{1}{2} + [E(x)+5]\cdot\frac{1}{3} + 2\cdot\frac{1}{6}$$

}

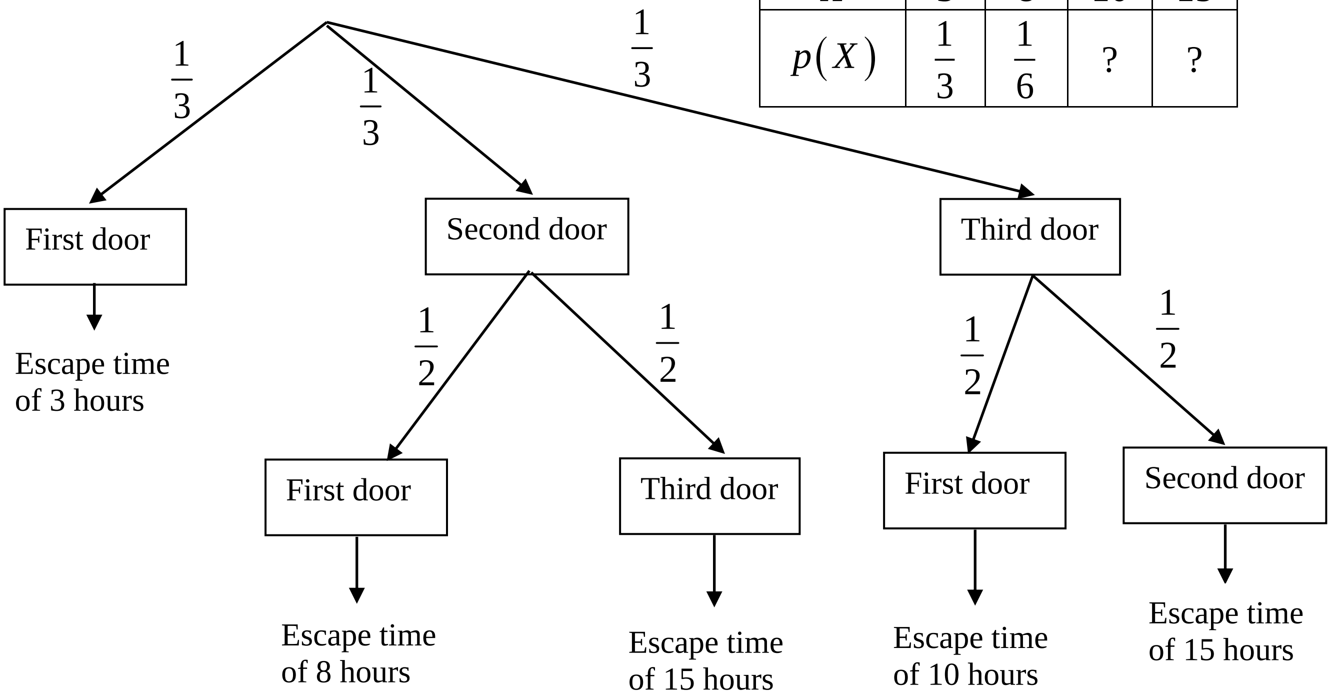
Working In A Coal Mine….

88. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours.

a) If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he escapes from the mine?

b) If we assume that the miner won't choose a door more than once, what is the expected length of time until he escapes from the mine?

X	3	8	10	15
$p(X)$	$\frac{1}{3}$	$\frac{1}{6}$	?	?



Consider a random variable X with the following probability distribution:

X	x_1	x_2	x_3
$p(x)$	p_1	p_2	p_3

Let Y be the number of trials of the experiment until the value x_2 *occurs. Let's find* $E(Y)$ *using conditional expectation.*

$$E(Y) = E[Y \mid X = x_2]P(X = x_2) + E[Y \mid X \neq x_2]P(X \neq x_2)$$
$$= 1 \cdot p_2 + [1 + E(Y)] \cdot (1 - p_2)$$

So we get the equation $p_2 E(Y) = 1$, *or* $E(Y) = \frac{1}{p_2}$. *So in general, the expected number of trials needed to get the value x, is* $\frac{1}{p(x)}$.

I'm In A Hurry, How Long Can I Expect To Wait?

89. a) Use the previous result to find the expected number of rolls of a fair die to get a 3.

b) Use the previous result to find the expected number of tosses of a fair coin to get tails.

c) If a card is randomly drawn from a standard 52 card deck, use the previous result to find the expected number of draws to get an ace. (The card is replaced after each draw.)

d) Use the previous result to find the expected number of tosses of three fair coins to get all three coins showing tails.

e) If two cards are randomly drawn without replacement from a standard 52 card deck, use the previous result to find the expected number of draws of two cards to get a pair of aces. (The two cards are replaced after each draw.)

Congratulations! You've Won A ToyYoda!

90. Sometimes the things you are collecting don't have the same probability of occurring. Suppose that a cereal company puts two different types of toys in its cereal boxes.

a) A Darth Vader toy occurs with probability of .8, and a Yoda toy occurs with probability of .2. What's the expected number of boxes of cereal you must buy to get a complete set of the two toys?

{Hint:

$$\begin{aligned} E(\text{\# of boxes}) &= E[\text{\# of boxes} \mid \text{first is Darth Vader}]P(\text{first is Darth Vader}) \\ &\quad + E[\text{\# of boxes} \mid \text{first is Yoda}]P(\text{first is Yoda}) \\ &= \left(1+\tfrac{1}{.2}\right)(.8)+\left(1+\tfrac{1}{.8}\right)(.2) \end{aligned}$$

.}

b) A Darth Vader toy occurs with probability of .75, and a Yoda toy occurs with probability of .25. What's the expected number of boxes of cereal you must buy to get a complete set of the two toys?

Flipped Off Twice.

91. a) An experiment has two phases: First 4 fair coins are flipped. Second, the coins that showed heads are removed, and the remaining coins are flipped again. What is the expected number of tails that occur in the second round of flipping?

{Hint:

$$\begin{aligned} E[\#\,of\ tails] &= E[\#\,of\ tails \mid no\ heads]\cdot P(no\ heads) + E[\#\,of\ tails \mid 1\,head]\cdot P(1\,head) + \\ &\quad \cdots + E[\#\,of\ tails \mid 4\,heads]\cdot P(4\,heads) \end{aligned}$$

}

b) Do the same as a), except now there are 20 fair coins.

{Hint:

$$\begin{aligned} E[\#\,of\ tails] &= E[\#\,of\ tails \mid no\ heads]\cdot P(no\ heads) + E[\#\,of\ tails \mid 1\,head]\cdot P(1\,head) + \\ &\quad \cdots + E[\#\,of\ tails \mid 20\,heads]\cdot P(20\,heads) \end{aligned}$$

}

Having Children Expectedly.

92. Suppose that in a faraway country, 20% of the families have no children, 20% have exactly 1 child, 30% have exactly 2 children, 20% have exactly 3 children, 10% have exactly 4 children, and no families have more than 4 children. If a child is chosen at random, what is the expected number of siblings of the child?

{Hint: In order to select a child, the child had to come from a family with at least 1 child. If there were 100 families in the country, then there would be 20 children from a 1 child family, 60 children from a 2 child family, 60 children from a 3 child family, and 40 children from a 4 child family. This gives a total of 180 children. If X is the number of siblings of a randomly selected child, then

X	0	1	2	3
P(X)	$\frac{20}{180}$	$\frac{60}{180}$	$\frac{60}{180}$	$\frac{40}{180}$

}

Oh, Darn, I Guess We'll Have To Keep On Trying!

93. What is the expected number of children that a couple must have in order to have both a boy and a girl?

{Hint: Here is a table of results and probabilities:

X	***2***	***3***	***4***	$\cdots$
Results	***BG,GB***	***BBG,GGB***	***BBBG,GGGB***	$\cdots$
P(X)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\cdots$

$$\begin{aligned} E(X) &= 2\cdot\tfrac{1}{2}+3\cdot\tfrac{1}{4}+4\cdot\tfrac{1}{8}+5\cdot\tfrac{1}{16}+\cdots \\ &= \tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\cdots \\ &\quad+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\cdots \\ &\quad+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\cdots \\ &\quad+\tfrac{1}{8}+\tfrac{1}{16}+\cdots \\ &\quad+\tfrac{1}{16}+\cdots \\ &= 1+1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\cdots \end{aligned}$$

}

Tenus Anyone?

94. Tim and Allen are playing a match of tenus. In a match of tenus, the two players play a series of games, each of which is won by one of the two players. The match ends when one player has won exactly two more games than the other player, at which point the player who has won more games wins the match. In odd-numbered games, Tim wins with probability $\frac{3}{4}$, and in the even-numbered games, Allen wins with probability $\frac{3}{4}$. What is the expected number of games in a match?

{Hint:

$$\begin{aligned} E[\text{\# of games}] &= E[\text{\# of games}|\text{TT}]\cdot P(TT) + E[\text{\# of games}|\text{TA}]\cdot P(TA) \\ &\quad + E[\text{\# of games}|\text{AT}]\cdot P(AT) + E[\text{\# of games}|\text{AA}]\cdot P(AA) \\ &= 2\cdot P(TT) + (2 + E[\text{\# of games}])\cdot P(TA) + (2 + E[\text{\# of games}])\cdot P(AT) \\ &\quad + 2\cdot P(AA) \end{aligned}$$

}

Don't Jack With Me!

95. In a card game, we remove the Jacks, Queens, Kings, and Aces from a deck of ordinary cards and shuffle them. You draw one of these cards. If it is an Ace, you are paid a dollar and the game is repeated. If it is a Jack, you are paid two dollars and the game ends; if it is a Queen, you are paid three dollars and the game is repeated; and if it is a King, you are paid four dollars and the game is repeated. What is the expected amount of money you will win from playing this game?

{Hint:

$$\begin{aligned} E[\text{winnings}] &= E[\text{winnings}|\text{A}]\cdot P(A) + E[\text{winnings}|\text{J}]\cdot P(J) \\ &\quad + E[\text{winnings}|\text{Q}]\cdot P(Q) + E[\text{winnings}|\text{K}]\cdot P(K) \\ &= (1 + E[\text{winnings}])\cdot\frac{1}{4} + 2\cdot\frac{1}{4} + (3 + E[winnings])\cdot\frac{1}{4} \\ &\quad + (4 + E[winnings])\cdot\frac{1}{4} \end{aligned}$$

}

Suppose there are 3 different kinds of trading cards, and when you buy a box of cereal, the probability that the box contains any one of the 3 kinds is the same.

The expected number of boxes you need to buy to get any one of the cards is 1.

The expected number of boxes you need to buy to get two different kinds of cards is 1 + (the expected number of boxes to get a different card). From the previous discussion, this is equal to $1+\frac{1}{\frac{2}{3}}=1+\frac{3}{2}=3\left(\frac{1}{2}+\frac{1}{3}\right)$. *The expected number of boxes you need to buy to get three different kinds of trading cards is 1 + (the expected number of boxes to get a different card) + (the expected number of boxes to get a 2nd different card). Again, from the previous discussion, this is equal to* $1+\frac{1}{\frac{2}{3}}+\frac{1}{\frac{1}{3}}=1+\frac{3}{2}+3=3\left(1+\frac{1}{2}+\frac{1}{3}\right)$. *In general with n different kinds of trading cards with equal probabilities, the expected number of boxes to get a complete set is* $n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$.

How Long Until I Get A Complete Set?

96. a) Use the previous discussion to find the expected number of rolls of a fair die to get all six faces to appear.

b) Do the same for a fair eight-sided die.

c) If you randomly draw a card from a standard 52 card deck, look at it, and then replace it and repeat, what is the expected number of draws until you see every one of the 52 cards?

There are special kinds of sums called geometric series. They are sums of the form $s_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$, where the fixed value r is called the common ratio. Here is a nice trick for finding a convenient formula for the sum of a geometric series:

$$\begin{aligned} s_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ -rs_n &= \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \\ \hline (1-r)s_n &= a(1-r^n) \end{aligned}$$

So if $r \neq 1$ then $s_n = \frac{a(1-r^n)}{1-r}$. If $r = 1$, then $s_n = \underbrace{a + a + a + a + \cdots + a}_{\text{n times}} = na$. So $s_n = \begin{cases} \frac{a(1-r^n)}{1-r} & ; r \neq 1 \\ na & ; r = 1 \end{cases}$. If $|r| < 1$, then we can assign a meaning to the infinite sum $s_\infty = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n + ar^{n+1} + \cdots$ since in this case r^n will die off to zero as n grows indefinitely. So we say that $s_\infty = \frac{a}{1-r}$.

The St. Peterhortonsburg Paradox.

97. A game is played in the following way: You flip a fair coin; if you see tails, you flip again, and the game continues until you see a head, which ends the game. If you see heads on the first flip, you receive d_1 dollars. If you see heads on the second flip, you receive d_2 dollars, and so on. A probability distribution for the money won in this game is the following:

X	d_1	d_2	d_3	d_4	d_5	d_6	$\cdots$	d_n	$\cdots$
$P(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

a) Use the previous geometric series result to show that the probability distribution is valid, i. e. the sum of all the probabilities is 1.

b) If the amount of money won starts at \$1 and increases by 1 each time, determine the expected value of your winnings.

X	1	2	3	4	5	6	$\cdots$	n	$\cdots$
$P(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

c) If the amount of money won is \$2 and increases by 1 each time, determine the expected value of your winnings.

X	2	3	4	5	6	7	$\cdots$	$n+1$	$\cdots$
$P(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

d) If the amount of money won starts at \$2 and then doubles each time after, determine the expected value of your winnings.

X	2	4	8	16	32	64	$\cdots$	2^n	$\cdots$
$P(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

e) If the amount of money won is as described in the table, determine if the expected value of your winnings is less than the expected value In part d).

X	$\frac{2}{1}$	$\frac{4}{2}$	$\frac{8}{3}$	$\frac{16}{4}$	$\frac{32}{5}$	$\frac{64}{6}$	$\cdots$	$\frac{2^n}{n}$	$\cdots$
$P(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

{Hint: $1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>2\cdot\frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>4\cdot\frac{1}{8}}+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{>8\cdot\frac{1}{16}}+\cdots$, *so*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots.\}$$

A newspaper carrier buys newspapers for 5 cents and sells them for 10 cents. She is given 3 cents the following day for each newspaper which is not sold. The carrier decides to predict how many newspapers she is going to sell to maximize her long-term profit. After studying what happens over a 100 day period, and taking into account the demand (not just the number of newspapers sold), she compiles the following table and derives probabilities from it:

# of newspaper in demand	# of days	cumulative # of days
0	0	0
1	1	1
2	0	1
3	2	3
4	1	4
5	3	7
6	2	9
7	1	10
8	1	11
9	1	12
10	3	15
11	1	16
12	2	18
13	4	22
14	1	23
15	3	26
16	2	28
17	3	31
18	4	35
19	4	39
20	4	43
21	6	49
22	2	51
23	4	55
24	3	58
25	4	62
26	6	68
27	3	71
28	7	78
29	5	83
30	4	87
31	3	90
32	4	94
33	4	98
34	1	99
35	0	99
36	1	100

For example, the probability of the demand being 4 newspapers per day is taken to be $\frac{1}{100}$, the probability of a demand of 17 is $\frac{3}{100}$, and the probability of a demand of 26 is $\frac{6}{100}$. Using these probabilities, the carrier computes the expected profits for each of the numbers 0 through 36. Her profit equals her revenue minus her cost, the number of newspapers she sold times ten cents plus the number of unsold newspapers times 3 cents minus the number of newspapers she bought times 5 cents.

For example, if she buys 6 newspapers, she observes from the cumulative column that the probability of selling 5 or fewer newspapers is $\frac{7}{100}$ so that the probability of selling 6 or more, therefore all 6 newspapers , is $\frac{93}{100}$. So her expected profit in this case is

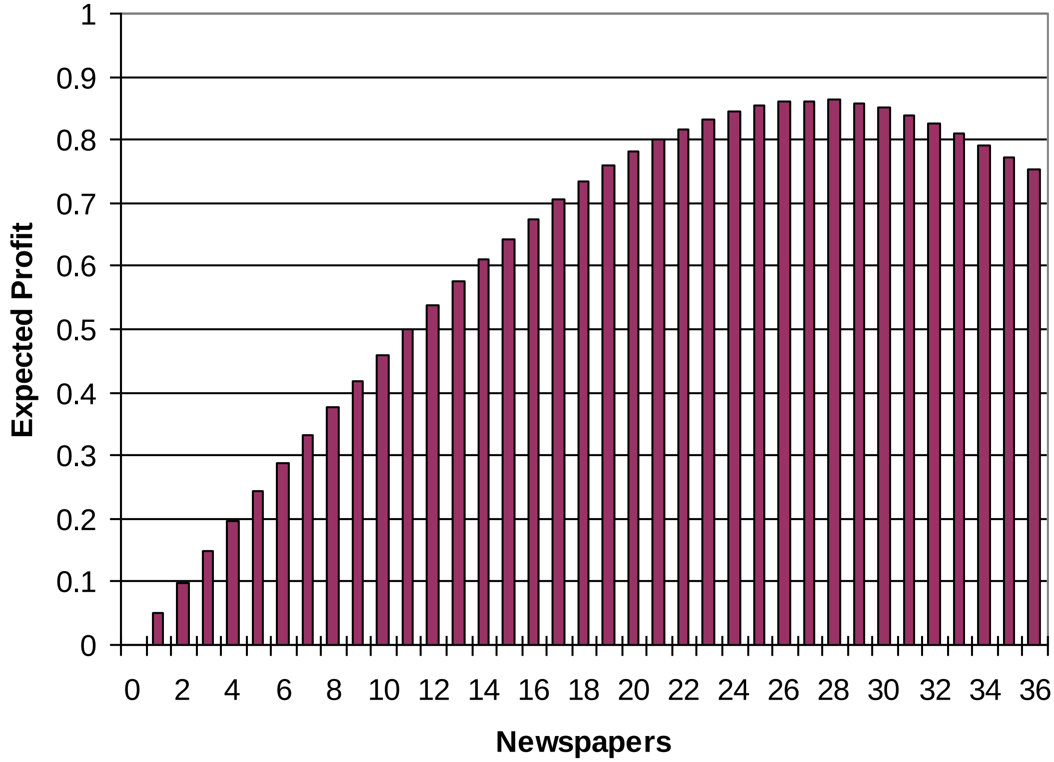
$$
\begin{gathered}
(\text{profit from } 6)\cdot(\text{probability of selling } 6)+(\text{profit from } 5)\cdot(\text{probability of selling } 5)\\
+(\text{profit from } 4)\cdot(\text{probability of selling } 4)+(\text{profit from } 3)\cdot(\text{probability of selling } 3)\\
+(\text{profit from } 2)\cdot(\text{probability of selling } 2)+(\text{profit from } 1)\cdot(\text{probability of selling } 1)\\
+(\text{profit from } 0)\cdot(\text{probability of selling } 0)\\
=\underbrace{30\cdot\frac{93}{100}}_{\text{6 or more}}+\underbrace{23\cdot\frac{3}{100}}_{5}+\underbrace{16\cdot\frac{1}{100}}_{4}+\underbrace{9\cdot\frac{2}{100}}_{3}+\underbrace{2\cdot\frac{0}{100}}_{2}+\underbrace{(-5)\cdot\frac{1}{100}}_{1}+\underbrace{(-12)\cdot\frac{0}{100}}_{0}\\
=28.88
\end{gathered}
$$

As another example, if she buys 10 newspapers, notice that the probability of her selling fewer than 10 newspapers is $\frac{12}{100}$, so the probability of her selling 10 or more, therefore all 10 newspapers, is $\frac{88}{100}$. Her expected profit in this case is given by

$$
\begin{gathered}
\underbrace{50\cdot\frac{88}{100}}_{\text{10 or more}}+\underbrace{43\cdot\frac{1}{100}}_{9}+\underbrace{36\cdot\frac{1}{100}}_{8}+\underbrace{29\cdot\frac{1}{100}}_{7}+\underbrace{22\cdot\frac{2}{100}}_{6}+\underbrace{15\cdot\frac{3}{100}}_{5}+\underbrace{8\cdot\frac{1}{100}}_{4}+\underset{3}{1\cdot\frac{2}{100}}+\underbrace{(-6)\cdot\frac{0}{100}}_{2}\\
+\underbrace{(-13)\cdot\frac{1}{100}}_{1}+\underbrace{(-20)\cdot\frac{0}{100}}_{0}\\
=45.94
\end{gathered}
$$

After spending a lot of time with these tedious calculations, the carrier compiles the following table and graph:

# of newspaper in demand	# of days	cumulative # of days	Expected profit
0	0	0	.0000
1	1	1	.0500
2	0	1	.0993
3	2	3	.1486
4	1	4	.1965
5	3	7	.2437
6	2	9	.2888
7	1	10	.3325
8	1	11	.3755
9	1	12	.4178
10	3	15	.4594
11	1	16	.4989
12	2	18	.5377
13	4	22	.5751
14	1	23	.6097
15	3	26	.6436
16	2	28	.6754
17	3	31	.7058
18	4	35	.7341
19	4	39	.7596
20	4	43	.7823
21	6	49	.8022
22	2	51	.8179
23	4	55	.8322
24	3	58	.8437
25	4	62	.8531
26	6	68	.8597
27	3	71	.8621
28	7	78	**.8624**
29	5	83	.8578
30	4	87	.8497
31	3	90	.8388
32	4	94	.8258
33	4	98	.8100
34	1	99	.7914
35	0	99	.7721
36	1	100	.7528



She determines that she should purchase 28 newspapers to maximize her expected profit. The carrier could have saved a lot of time and effort if she would have looked at the problem in the following way:

Let x be a whole number from 0 to 36, $0 \le x \le 36$. Let $P(x)$ be the probability of selling x or fewer newspapers. These probabilities can be read directly from the cumulative column in the original table. For example, $P(10) = \frac{15}{100}$ and $P(17) = \frac{31}{100}$. Now suppose that she orders x newspapers and considers what would happen if one more newspaper were ordered. On the additional newspaper she would make 5 cents with probability of $1 - P(x)$ and would lose 2 cents with probability $P(x)$. So her expected profit on the additional newspaper would be

$$5 \cdot \left[1 - P(x)\right] - 2 \cdot P(x)$$

or

$$5 - 7 \cdot P(x).$$

An additional newspaper should be purchased if $5 - 7 \cdot P(x) > 0$, which means that an additional newspaper should be purchased if $P(x) < \frac{5}{7} \approx .714$. From the table, $P(27) = .71$ and $P(28) = .78$. Since $P(27) < \frac{5}{7} < P(28)$, one additional newspaper beyond 27 should be purchased. This means that the maximum expected profit occurs at 28 newspapers.

Extra! Extra!

98. a) Suppose the newspaper company changes its policy and gives only 2 cents the following day for each newspaper which is not sold. How many newspapers should the carrier purchase to maximize her expected profit?

b) Suppose the newspaper company changes its policy and no money is given back the following day for each newspaper which is not sold. How many newspapers should the carrier purchase to maximize her expected profit?

Let's consider the following game: Two players A and B sit across a table from each other. Each has a coin. For 100 times, the coins are going to be placed simultaneously on the table with the following payoff rules(T for tails and H for heads):

A shows	**B shows**	
H	H	B pays A $3
H	T	A pays B $1
T	H	A pays B $6
T	T	B pays A $4

It is standard practice to put such a game into matrix form. That is we tabulate the above information as a two by two matrix as follows:

$$\begin{array}{cc} & \text{B} \\ & \begin{array}{cc} \text{H} & \text{T} \end{array} \\ \text{A} \begin{array}{c} \text{H} \\ \text{T} \end{array} & \begin{bmatrix} 3 & -1 \\ -6 & 4 \end{bmatrix} \end{array}$$

All of the entries in this matrix represent payments or losses to player A. For example, the 3 in the first row, first column, indicates a payment of $3 to player A if both players show Heads, while the – 1 in the first row, second column indicates a loss of $1 to player A.

Let's suppose that player A decides to show Heads $\frac{1}{4}$ of the time and tails $\frac{3}{4}$ of the time, while player B decides to show Heads $\frac{1}{2}$ of the time and Tails $\frac{1}{2}$ of the time. This information is recorded in the updated matrix below:

$$
\begin{array}{cc} & \text{B} \\ & \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \text{H} & \text{T} \end{array} \\ \text{A} \begin{array}{cc} \frac{1}{4} & \text{H} \\ \frac{3}{4} & \text{T} \end{array} & \begin{bmatrix} 3 & -1 \\ -6 & 4 \end{bmatrix} \end{array}
$$

Let's compute the expected winnings(losses) of player A by first compiling a table of possible values of A's winnings along with their respective probabilities.

A's winnings	Probability
3	$\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$
-1	$\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$
-6	$\frac{3}{4}\cdot\frac{1}{2}=\frac{3}{8}$
4	$\frac{3}{4}\cdot\frac{1}{2}=\frac{3}{8}$

So the expected winnings of player A is $3\cdot\frac{1}{8}+(-1)\cdot\frac{1}{8}+(-6)\cdot\frac{3}{8}+4\cdot\frac{3}{8}=-\frac{4}{8}=-\$.50$. If player A and player B continue to play this strategy, then A's long-term average winnings will be $-\$.50$ per play of the game. The expected winnings of player B is $(-3)\cdot\frac{1}{8}+1\cdot\frac{1}{8}+6\cdot\frac{3}{8}+(-4)\cdot\frac{3}{8}=\frac{4}{8}=\$.50$, the negative of player A's. This will be the case in general.

Let's suppose that player A decides to show Heads p of the time, where p is a number from 0 to 1, $0\le p\le 1$. Then Tails will be shown $1-p$ of the time. Let's also suppose that player B will show Heads q of the time and Tails $1-q$ of the time, with again, $0\le q\le 1$. These are indicated in the following diagram:

$$
\begin{array}{cc} & \text{B} \\ & \begin{array}{cc} q & 1-q \\ \text{H} & \text{T} \end{array} \\ \text{A} \begin{array}{cc} p & \text{H} \\ 1-p & \text{T} \end{array} & \begin{bmatrix} 3 & -1 \\ -6 & 4 \end{bmatrix} \end{array}
$$

The expected gain G to player A is given by

$$\begin{aligned} G &= 3(pq) + (-1)p(1-q) + (-6)q(1-p) + 4(1-p)(1-q) \\ &= 14pq - 5p - 10q + 4 \\ &= 14\left(pq - \tfrac{5}{14}p - \tfrac{10}{14}q\right) + 4 \\ &= 14\left(p - \tfrac{10}{14}\right)\left(q - \tfrac{5}{14}\right) + \tfrac{6}{14} \end{aligned}$$

So $G = 14\left(p - \frac{10}{14}\right)\left(q - \frac{5}{14}\right) + \frac{6}{14}$. From this we see that if player A shows Heads $\frac{10}{14}$ of the time, then no matter what player B does, player A can expect to gain an average of $\$\frac{6}{14}$ per play. If player A chooses $p > \frac{10}{14}$, making $p - \frac{10}{14} > 0$, and player B becomes aware of it, then player B could take $q < \frac{5}{14}$, making $q - \frac{5}{14} < 0$ and detract from the $\$\frac{6}{14}$ gain for player A. In fact, player B could choose $q < \frac{5}{14}$ so that the gain for player A turns into a loss for player A. We say that the optimal strategy for player A is to choose $p = \frac{10}{14} = \frac{5}{7}$. From player B's perspective, his expected gain is the negative of the expected gain for player A, $-14\left(p - \frac{10}{14}\right)\left(q - \frac{5}{14}\right) - \frac{6}{14}$. player B is in trouble. If player A chooses $p = \frac{10}{14}$, then no matter what player B does, he can expect to lose $\$\frac{6}{14}$. If player B decides to play this game and choose $q > \frac{5}{14}$, making $q - \frac{5}{14} > 0$, and player A becomes aware of it, then player A can choose $p > \frac{10}{14}$, making $p - \frac{10}{14} > 0$, and player B will lose more than $\$\frac{6}{14}$ per play. Similarly, if player B chooses $q < \frac{5}{14}$, making $q - \frac{5}{14} < 0$, then player A could choose $p < \frac{10}{14}$, making $p - \frac{10}{14} < 0$, and again player B can expect to lose more than $\$\frac{6}{14}$ per play. If player B decides to play the game, then his optimal strategy is to choose $q = \frac{5}{14}$.

Let's look at another example:

$$\begin{array}{ccc} & & \text{B} \\ & & \begin{array}{cc} \text{H} & \text{T} \end{array} \\ \text{A} & \begin{array}{c} \text{H} \\ \text{T} \end{array} & \begin{bmatrix} -7 & 3 \\ -6 & 8 \end{bmatrix} \end{array}$$

In this case, the expected gain for player A is given by

$$\begin{aligned} G &= (-7)(pq) + 3p(1-q) + (-6)q(1-p) + 8(1-p)(1-q) \\ &= 4pq - 5p - 14q + 8 \\ &= 4\left(pq - \tfrac{5}{4}p - \tfrac{14}{4}q\right) + 8 \\ &= 4\left(p - \tfrac{14}{4}\right)\left(q - \tfrac{5}{4}\right) - \tfrac{19}{2} \end{aligned}$$

From player A's perspective, no matter how player B chooses $0 \le q \le 1$, $q - \dfrac{5}{4}$ will be negative, so player A needs to make $p - \dfrac{14}{4}$ as negative as possible to minimize the loss. The optimal strategy for player A is to choose $p = 0$. From player B's perspective, his expected gain is the negative of player A's, $-4\left(p - \tfrac{14}{4}\right)\left(q - \tfrac{5}{4}\right) + \tfrac{19}{2}$. No matter how player A chooses $0 \le p \le 1$, $-4\left(p - \dfrac{14}{4}\right)$ will be positive, so player B needs to make $q - \dfrac{5}{4}$ as little negative as possible to maximize the gain. The optimal strategy for player B is to choose $q = 1$.

A strategy in which you choose different options is called a mixed strategy. The first example is a mixed strategy. A strategy in which you always choose the same option is called a pure strategy. The second example is a pure strategy since A always chooses Tails and B always chooses Heads.

I’ll Show You Mine When You Show Me Yours.

99. Determine optimal strategies for players A and B if

a)

$$A \begin{array}{c} \\ H \\ T \end{array} \overset{\displaystyle B}{\begin{array}{cc} H & T \\ \end{array}} \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$$

b)

$$A \begin{array}{c} H \\ T \end{array} \overset{\displaystyle B \atop H \quad T}{\begin{bmatrix} 3 & -4 \\ -6 & 7 \end{bmatrix}}$$

c)

$$A \begin{array}{c} H \\ T \end{array} \overset{\displaystyle B \atop H \quad T}{\begin{bmatrix} 6 & 1 \\ -3 & -7 \end{bmatrix}}$$

d)

$$A \begin{array}{c} H \\ T \end{array} \overset{\displaystyle B \atop H \quad T}{\begin{bmatrix} -4 & -5 \\ 10 & -6 \end{bmatrix}}$$

Consider the following war between two small countries A and B. We assume the following:

1) Country A has two planes, and there are two air routes from A to B. In country B there is a small bridge which is vital to B’s military efforts. The two planes of country A are to be used to destroy the bridge. The bridge requires about 24 hours to rebuild and each plane makes one daily flight in an attempt to keep the bridge in an unusable condition. If a plane is shot down, a large neutral power will immediately supply country A with a new plane.

2) Country B has two anti-aircraft guns which it uses along the air routes in an attempt to shoot down the planes from country A.

3) As there are two routes from A to B, country A can send both planes along one route or one plane along each route.

4) Country B can place both guns along one route or one gun along each route.

5) If one plane(two planes) travel(s) along a route on which there is one gun(two guns), then that plane(both planes) will be shot down. However if the two planes travel along a route on which there is only one gun, then only one plane will be shot down.

6) If a plane gets through to the bridge, then the bridge will be destroyed.

War Games.

100. Let D stand for using different routes and S stand for using the same route. Here is a table showing the results:

A	**B**	**Probability that bridge is destroyed**
D	D	0
D	S	1
S	D	1
S	S	½

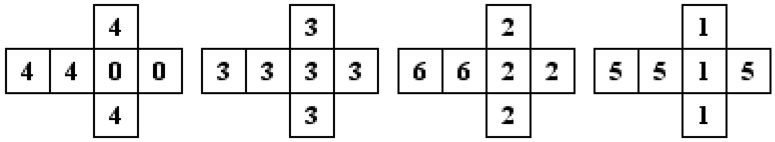
Here's the matrix for this game:

$$\begin{array}{cc} & \text{B} \\ & \begin{array}{cc} \text{D} & \text{S} \end{array} \\ \text{A} \quad \begin{array}{c} \text{D} \\ \text{S} \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \end{array}$$

Find optimal strategies for countries A and B. Remember that country A wants the probability high for destroying the bridge, while country B wants it low.

Die! You Intransitive Caster.

101. Here are four interesting and famous dice:



The game is as follows: The first player chooses a die, and the second player chooses one of the three remaining dice. The two dice are rolled, and the higher number wins. Show that no matter which die the first player chooses, if the second player chooses the die to its left in the above diagram, the second player will win with probability $\frac{2}{3}$. *(If the first player chooses the far left die, then the second player chooses the far right die.)*

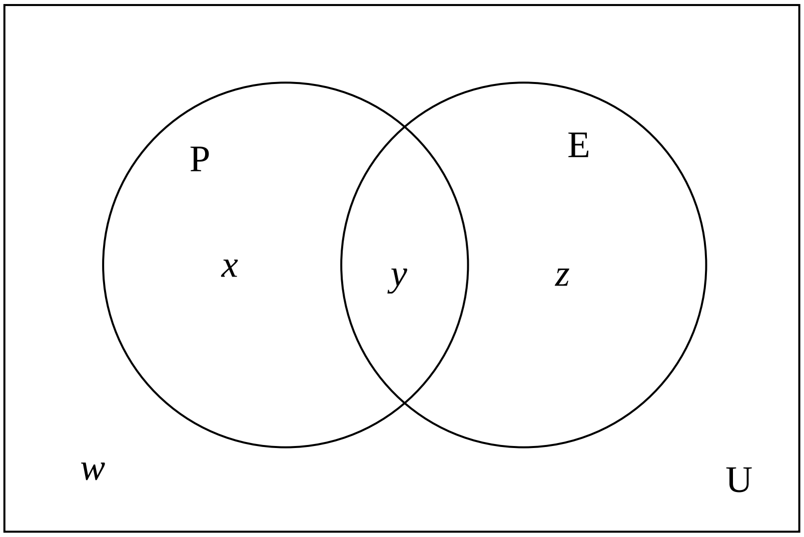
{Hint: For the first two dice, here is the table of possible outcomes:

	4	***4***	***4***	***4***	***0***	***0***
3	*W*	*W*	*W*	*W*	*L*	*L*
3	*W*	*W*	*W*	*W*	*L*	*L*
3	*W*	*W*	*W*	*W*	*L*	*L*
3	*W*	*W*	*W*	*W*	*L*	*L*
3	*W*	*W*	*W*	*W*	*L*	*L*
3	*W*	*W*	*W*	*W*	*L*	*L*

Notice that the left die wins with probability $\frac{2}{3}$.*}*

How Much Is A Picture Worth?

102. Mary is taking two courses, photography and economics. Student records indicate that the probability of passing photography is .75, that of failing economics is .65, and that of passing at least one of the two courses is .85.



Find the probability of the following:

a) Mary will pass economics.

b) Mary will pass both courses.

c) Mary will fail both courses.

d) Mary will pass exactly one course.

{Hint: $x+y=.75, x+w=.65, x+y+z=.85, x+y+z+w=1.$*}*

Roulette is the oldest casino game still being played. Its invention has variously been credited to the French mathematician Blaise Pascal, the ancient Chinese, a French monk, and the Italian mathematician Don Pasquale. It became popular when a French policeman introduced it to Paris in 1765 in an attempt to take the advantage away from dishonest gamblers. When the Monte Carlo casino opened in 1863, roulette was the most popular game, especially among the aristocracy.

The American roulette wheel has 38 compartments. Thirty-six compartments are numbered 1 through 36 with 18 colored red and the other 18 colored black. The remaining two are numbered 0 and 00 and are colored green. A martingale is a gambling strategy in which the gambler doubles his or her bet after each loss. A person using this strategy with this roulette wheel, concentrating on the black-number bet, might lose three times before winning. This strategy would result in a net gain as in the following table:

Bet Number	Bet	Result	Total Winnings
1	$1	Lose	-$1
2	$2	Lose	-$3
3	$4	Lose	-$7
4	$8	Win	$1

This seems to be a great strategy. Sooner or later the player will win a bet, and because each bet is larger than the player's total losses, he or she has to come out ahead! The only problem is that the player might go bankrupt before winning a bet!

Let's call a round of martingale betting in roulette a consecutive sequence of losses followed by a win or bankruptcy of the player. In roulette with a black-number bet, the probability of losing is $\frac{20}{38}=\frac{10}{19}$. If the maximum number of losses the player can afford is *n*, then the probability of having *n* consecutive losses is $\left(\frac{10}{19}\right)^n$, and the probability of not having *n* consecutive losses is $1-\left(\frac{10}{19}\right)^n$. With an opening bet of $1, the payoffs of a round are as follows:

Number of Losses	Payoff	Probability
$<n$	$1	$1-\left(\frac{10}{19}\right)^n$
n	$-\$\left(2^n-1\right)$	$\left(\frac{10}{19}\right)^n$

Wink Martingale.

103. a) Find the expected payoff of a single round of martingale betting on black-number only.

b) What happens to the expected payoff as the number of times the player can afford to lose increases?

c) If a player had $100, and started with a $1 bet, how many successive losses could the player afford? What would be the expected payoff of a single round for this player?

When It Comes To Rolling The Meatless Bones, Two Out Of Three Aint Bad.

104. Three fair dice are rolled.

a) What is the probability that the same number appears on exactly two of the three dice?

b) What is the probability that different numbers appear on the three dice?

Come Fly The Unfriendly Skies?

105. An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

Galileo, Go Figaro!

106. In Galileo's time, people thought that when three fair dice were rolled, a sum of 9 and a sum of 10 had the same probability since each could be obtained in 6 ways:

9: $1+2+6, 1+3+5, 1+4+4, 2+2+5, 2+3+4, 3+3+3$

10: $1+3+6, 1+4+5, 2+2+6, 2+3+5, 2+4+4, 3+3+4$

Compute the probabilities of these sums and determine which is more likely to occur.

How Big? How Small?

107. a) For two events, A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. This can be rearranged into $P(A \cap B) = P(A) + P(B) - P(A \cup B)$. Use the fact that $P(A \cup B) \le 1$, to write an inequality involving $P(A \cap B), P(A)$, and $P(B)$.

b) For two events, A and B, $P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$, so $P(A \cap B) \le \min\left[P(A), P(B)\right]$. For $P(A) = .4$ and $P(B) = .7$, use the inequality from part a) and the previous inequality to find the maximum and minimum possible values for $P(A \cap B)$.

Disputed Elections

In a close election in a small town, 2,656 people voted for candidate A compared to 2,594 who voted for candidate B, a margin of victory of 62 votes. An investigation of the election, instigated no doubt by the loser, found that 136 of the people who voted in the election should not have. Since this is more than the margin of victory, should the election results be thrown out even though there was no evidence of fraud on the part of the winner's supporters?

Like many problems that come from the real world (DeMartini v. Power, 262 NE2d 857), this one is not precisely formulated. To turn this into a probability problem, we suppose that all the votes were equally likely to be one of the 136 erroneously cast and we investigate what happens when we remove 136 marbles from a bowl with 2,656 white marbles and 2,594 black marbles. Now the probability of removing exactly m white and $136-m$ black marbles is

$$\frac{{}_{2656}C_m \cdot {}_{2594}C_{136-m}}{{}_{5250}C_{136}}.$$

In order to reverse the election, we must have $2656-m \le 2594-(136-m)$ or that $m \ge 99$.

$$P(m \ge 99) = P(m=99) + P(m=100) + \cdots + P(m=136)$$

$$= \frac{{}_{2656}C_{99} \cdot {}_{2594}C_{136-99}}{{}_{5250}C_{136}} + \frac{{}_{2656}C_{100} \cdot {}_{2594}C_{136-100}}{{}_{5250}C_{136}} + \cdots + \frac{{}_{2656}C_{136} \cdot {}_{2594}C_{136-136}}{{}_{5250}C_{136}}.$$

Using Excel, we can conclude that the probability that the removal of the 136 randomly chosen votes will reverse the election is about .00000007492. This computation supports the Court of Appeals decision to overturn a lower court ruling that voided the election in this case.

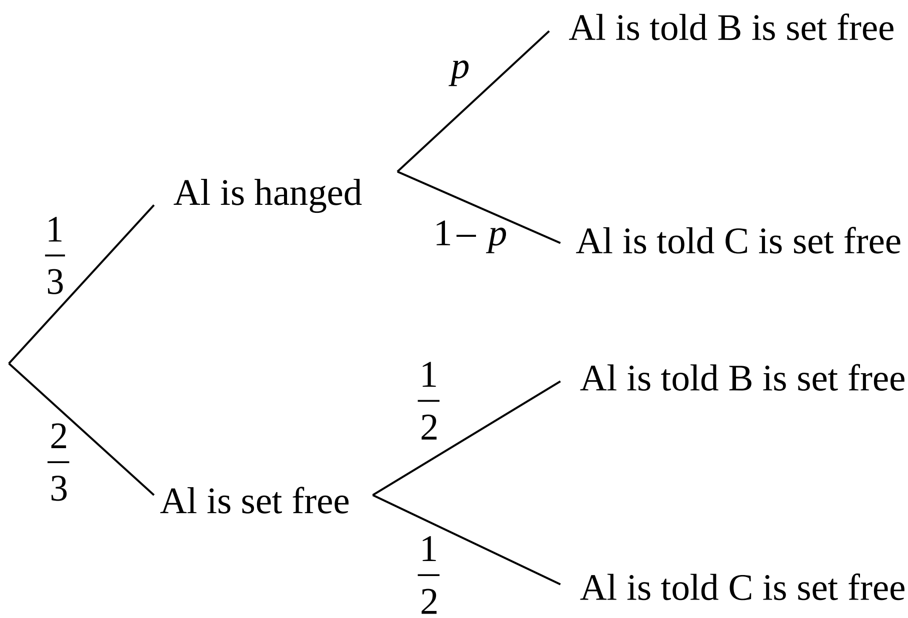
101 Hanging Chads.

108. In the case of Ipolito v. Power, 241 NE2d 232, the winning margin was 1,422 to 1,405, but 101 votes had to be thrown out. Compute the probability of the election being reversed with the removal of these 101 votes, and comment on whether you think the election results should be voided.

And Betty When You Call Me, You Can Call Me Al.

109. Three prisoners, Al, Bob, and Charlie, are in a cell. By a random selection, at dawn, two will be set free, and one will be hanged, but they don't know which one will be hanged. A guard named Betty offers to tell Al the name of one of the other prisoners who will go free, but Al stops her, screaming, "No, don't! That would increase my chances of being hanged from $\frac{1}{3}$ to $\frac{1}{2}$." Is Al correct?

{Hint:



You want $P(\text{Al is hanged}|\text{Al is told B is set free or Al is told C is set free})$.

Are You Always Late At The Airport?

110. You are going to meet a friend at the airport. Your experience tells you that the plane is late 70% of the time when it rains, but is only late 20% of the time when it doesn't rain. The weather forecast for that morning calls for a 40% chance of rain. What is the probability that the plane will be late?

"How Many Heads?", He Said Expectedly.

111. Suppose that a fair die is rolled, and then a fair coin is flipped that number of times. What is the expected number of Heads that will occur?

Hint: From the conditional expectation formula, you get

$$E[\text{\# of Heads}] = E[\text{\# of Heads}|\text{roll a 1}]\cdot P(\text{roll a 1}) + E[\text{\# of Heads}|\text{roll a 2}]\cdot P(\text{roll a 2}) + \cdots$$
$$+E[\text{\# of Heads}|\text{roll a 6}]\cdot P(\text{roll a 6})$$

"Tails!" He Said Repeatedly.

112. Suppose that a fair coin is flipped repeatedly until two consecutive tails occur. What is the expected number of flips required to get two consecutive tails?

Hint: From the conditional expectation formula, you get

$$\begin{aligned} E[\#\text{ of flips}] &= E[\#\text{ of flips}|\text{first flip is heads}]\cdot P(\text{first flip is heads}) \\ &\quad + E[\#\text{ of flips}|\text{first flip is tails}]\cdot P(\text{first flip is tails}) \\ &= (1+E[\#\text{ of flips}])\cdot P(\text{first flip is heads}) \\ &\quad + E[\#\text{ of flips}|\text{first two flips are TH}]\cdot P(\text{first two flips are TH}) \\ &\quad + E[\#\text{ of flips}|\text{first two flips are TT}]\cdot P(\text{first two flips are TT}) \\ &= (1+E[\#\text{ of flips}])\cdot P(\text{first flip is heads}) \\ &\quad (2+E[\#\text{ of flips}])\cdot P(\text{first two flips are TH}) \\ &\quad +2\cdot P(\text{first two flips are TT}) \end{aligned}$$

In the previous problem, without using conditional expectation, let's find the expected number of tosses of a fair coin needed to get two consecutive tails:

2, TT with probability $\frac{1}{4}$

3, HTT with probability $\frac{1}{8}$

4, THTT or HHTT with probability $\frac{2}{16}$

5, HHHTT or THHTT or HTHTT with probability $\frac{3}{32}$

6, HHHHTT or THHHTT or HTHHTT or HHTHTT or THTHTT with probability $\frac{5}{64}$

And so on.

So the expected number of tosses is

$$2\cdot\frac{1}{4}+3\cdot\frac{1}{8}+4\cdot\frac{2}{16}+5\cdot\frac{3}{32}+6\cdot\frac{5}{64}+\cdots$$

Where each term is a whole number starting with 2 times a fraction whose numerator is a Fibonacci number(1, 1, 2, 3, 5, 8, 13, 21, …) and whose denominator is a power of 2. There is a formula that generates the Fibonacci numbers: $F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$, so we can write the expected number of tosses as

$$2\cdot\frac{1}{4}+3\cdot\frac{1}{8}+4\cdot\frac{2}{16}+5\cdot\frac{3}{32}+6\cdot\frac{5}{64}+\cdots+(n+1)\cdot\frac{\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n}{2^{2n+1}\sqrt{5}}+\cdots$$

Or in short-hand notation as

$$\sum_{n=1}^{\infty}\left[(n+1)\cdot\frac{\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n}{2^{2n+1}\sqrt{5}}\right]=\frac{1}{2\sqrt{5}}\left[\sum_{n=1}^{\infty}(n+1)\cdot\left(\frac{1+\sqrt{5}}{4}\right)^n-\sum_{n=1}^{\infty}(n+1)\cdot\left(\frac{1-\sqrt{5}}{4}\right)^n\right]$$

Using calculus, you can show that $\sum_{n=1}^{\infty}(n+1)\cdot x^n=\frac{1}{(1-x)^2};-1<x<1$. So we get

$$\frac{1}{2\sqrt{5}}\left[\frac{1}{\left(1-\frac{1+\sqrt{5}}{4}\right)^2}-\frac{1}{\left(1-\frac{1-\sqrt{5}}{4}\right)^2}\right]=\frac{1}{2\sqrt{5}}\left[\left(3+\sqrt{5}\right)^2-\left(3-\sqrt{5}\right)^2\right]=\frac{12\sqrt{5}}{2\sqrt{5}}=6.$$

Homer Simpson's Paradox.

113. There are two boxes. Each box contains a mixture of plain and sprinkled donuts. Homer chooses a box, and then randomly chooses a donut from that box. If it's sprinkled, Homer wins, but if it's plain, Homer loses.

In Game #1: Box A has 5 sprinkled donuts and 6 plain donuts.
Box B has 3 sprinkled donuts and 4 plain donuts.

Since the probability of winning if Homer picks Box A is $\frac{5}{11}$, and the probability of winning if Homer picks Box B is $\frac{3}{7}$, Homer would of course pick Box A, since $\frac{5}{11}>\frac{3}{7}$.

In Game #2: Box C has 6 sprinkled donuts and 3 plain donuts.
Box D has 9 sprinkled donuts and 5 plain donuts.

Since the probability of winning if Homer picks Box C is $\frac{6}{9}$, and the probability of winning if Homer picks Box D is $\frac{9}{14}$, Homer would of course pick Box C, since $\frac{6}{9}>\frac{9}{14}$.

In Game #3: Boxes A and C are combined into Box E.
Boxes B and D are combined into Box F.

Which box, E or F should Homer pick to maximize his probability of getting a sprinkled donut? Does the answer surprise you? Explain.

Repeatedly Red.

114. A box has 10 red marbles and 8 blue marbles.

a) If you randomly remove 4 marbles, one at a time, without replacement, what is the probability that, at some point, you have chosen two consecutive marbles that are red?

Hint: There are $18 \cdot 17 \cdot 16 \cdot 15 = 73,440$ *different ways to remove the marbles from the box.*

Systematically count how many ways you won't have two consecutive red marbles:

No red marbles:

blue	*blue*	*blue*	*blue*

There are $8 \cdot 7 \cdot 6 \cdot 5 = 1,680$ *ways with no red marbles.*

One red marble:

red	*blue*	*blue*	*blue*
blue	*red*	*blue*	*blue*
blue	*blue*	*red*	*blue*
blue	*blue*	*blue*	*red*

There are $4 \cdot 10 \cdot 8 \cdot 7 \cdot 6 = 13,440$ *ways with one red marble.*

Two red marbles:

red	*blue*	*red*	*blue*
blue	*red*	*blue*	*red*

There are $2 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 10,080$ *ways with one red marble.*

Keep going, if necessary!

b) If you randomly remove 5 marbles, one at a time, without replacement, what is the probability that, at some point, you have chosen two consecutive marbles that are red?

Naughty Scotty.

115. Lucius, Sarah, Brandon, and Scott take Algebra together. On their first exam, Lucius got a 94, Sarah got a 91, Brandon got a 95, and Scott's score was strictly between 81 and 87. If the mean of all four of their scores is a whole number, what was Scott's score on the first exam? And what was the mean score for the first exam?

Gimme An A; Gimme A B!

116. Teams A and B play a series of games; whoever wins two games first wins the series. If Team A has a 70% chance of winning any single game, what is the probability that Team A wins the series?
{Hint: Team A will be the winner only if the following results occur:
AA, BAA, ABA.}

Together, Again!

117. A library shelf contains seven books. Three books are math books and four books are science books. In how many different ways can the books be arranged on the shelf so that all the math books will be together?

Three Numbers In The Middle Of The Street, If You Know What I Mean.

118. The mean of three numbers is 10, and their median is 10.3. If the difference between the largest and the smallest of the three numbers is 2, then what are the three numbers?

The Penultimate Adventure.

119. The mean of five numbers is 69. The median is 83, and the mode is 85. The range of the five numbers is 70. Find the second-smallest number.

They're All The Same, If You Know What I Mean.

120. If the mean, median, and mode are all equal for the data set $\{8,3,x,5,4\}$, then what is the value of x?

The Terminator.

121. If the numerator of a fraction is randomly selected from the set $\{1,3,5,7,9\}$, and the denominator is randomly selected from the set $\{1,2,3,4,5\}$, what is the probability that the decimal representation of the fraction is a terminating decimal?

	1	**3**	**5**	**7**	**9**
1	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{5}{1}$	$\frac{7}{1}$	$\frac{9}{1}$
2	$\frac{1}{2}$				
3	$\frac{1}{3}$				
4	$\frac{1}{4}$				
5	$\frac{1}{5}$				

It All Adds Up To Something.

122. a) There are 120 five-digit numbers that use all the digits 1 through 5 exactly once. What is the sum of the 120 numbers?

Hint:
$$\left.\begin{array}{r} 12345 \\ 12354 \\ \vdots \\ +\,54321 \\ \hline \end{array}\right\} \textit{120 numbers}$$
How many of each digit occur in each column?

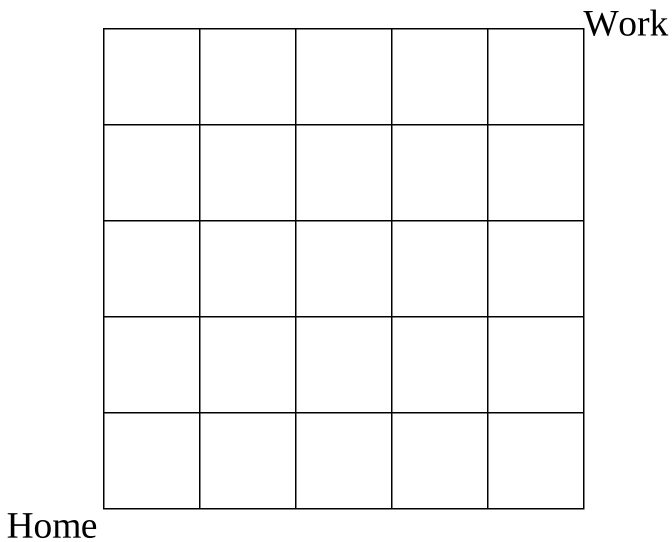
b) If the digits can be repeated, then there are 3,125 five-digit numbers that can be formed. What is the sum of the 3,125 numbers?

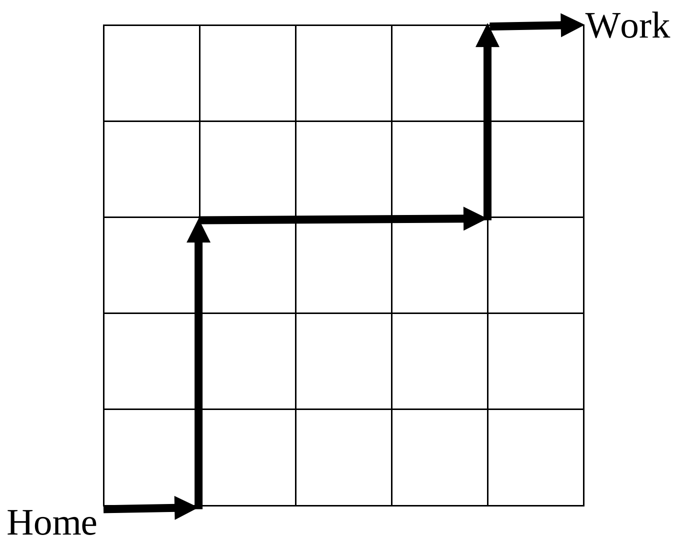
c) Repeat part a) with six-digit numbers using the digits 1 through 6.

d) Repeat part b) with six-digit numbers using the digits 1 through 6.

A Real Noreaster.

123. Ms. Jones likes to take a different route to work every day. She will quit her job the day she has to repeat her route. Her home and work are pictured in the grid of streets below. If she never backtracks(she only travels north or east), how many days will she work at this job?





{Hint: Each trip can be thought of as a permutation of a word with 5 E's and 5 N's.}

The Deadly Sin Of Seven.

124. Find the number of positive integers less than 100,000 that contain at least one digit of 7.
{Hint: How many positive integers are less than 100,000? How many of them don't have a digit of 7?}

Put Up Your Marbles And Box.

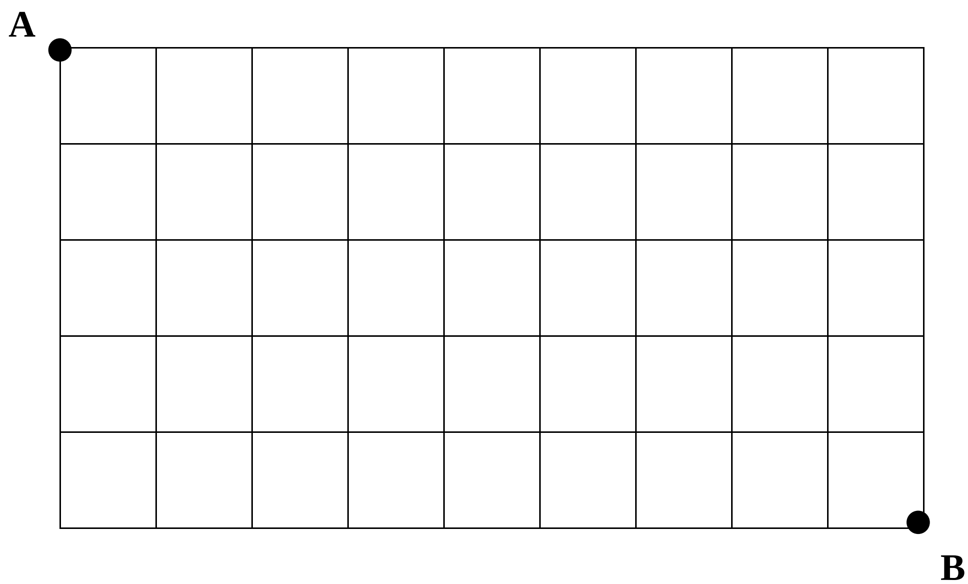
125. a) Find the number of ways that seven red marbles and eight white marbles can be placed into 3 boxes if each box contains at least one of each color.

b) Find the number of ways that seven red marbles and eight white marbles can be placed into 3 boxes if some of the boxes might not have each color or may be empty.

{Hint: See the hint for #5.}

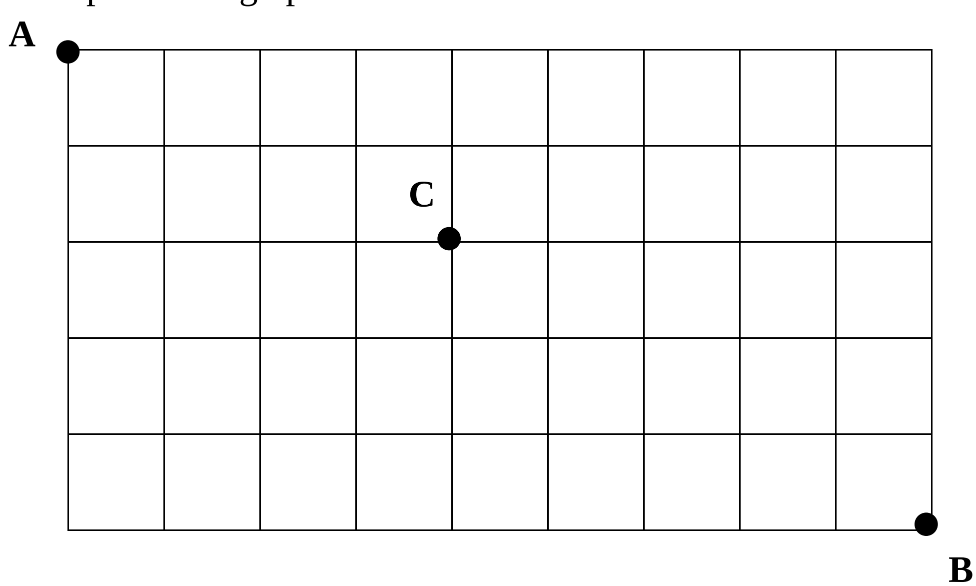
More Than One Way To Get From A To B.

126. a) How many paths are possible from A to B if all movements must be to the right or down?



{Hint: See the hint for #139.}

b) How many paths are possible from A to B if all movements must be to the right or down, and you must pass through point C?



Imelda Returns, Again.

127. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, without replacement, what is the probability that there will be

a) no complete pair?

Hint: Here are the ten pairs of shoes:

L_1R_1	L_2R_2	L_3R_3	L_4R_4	L_5R_5	L_6R_6	L_7R_7	L_8R_8	L_9R_9	$L_{10}R_{10}$

20	18	16	$\cdots$	6
Choices for 1st shoe	**Choices for 2nd shoe**	**Choices for 3rd shoe**	$\cdots$	**Choices for 8th shoe**

{This assumes order, so must use $n(S) = {}_{20}P_8$ *}*

b) exactly one complete pair?

Don't Pollute Your Mind.

128. In a study of water near power plants and other industrial plants that release wastewater into the water system, it was found that 5% showed signs of chemical and thermal pollution, 40% showed signs of chemical pollution, and 35% showed signs of thermal pollution.

a) What is the probability that a nearby stream that shows signs of thermal pollution will show signs of chemical pollution?

b) What is the probability that a nearby stream showing chemical pollution will not show signs of thermal pollution?

{Hint: conditional probability}

I Want A Retest.

129. Suppose that there is a cancer diagnostic test which is 95% accurate both on those that do and those that do not have the disease. If 4% of the population have cancer,

a) Find the probability that a person has cancer given that the test indicates that they do.

b) Find the probability that a person has cancer given that the test is taken twice and both times indicate that they do.

{Hint: Make a probability tree.}

c) Find the probability that a person has cancer given that the test is taken thrice and each time indicates that they do.

Designer Genes.

130. A particular genetic disorder occurs in .8% of the population. A test for the disorder can accurately detect it in 99.5% of those who have it, but this test gives a false positive result for 2% of those who do not have the disorder.

a) If the test indicates that you have the disorder, what is the probability that you have the disorder?

b) If you take the test twice and both times get a positive result, what is the probability that you have the disorder?

c) If the test indicates that you do not have the disorder, what is the probability that you do not have the disorder?

d) If you take the test twice and both times get a negative result, what is the probability that you not have the disorder?

{Hint: Make a probability tree.}

Are You Sure You Want To Go First?

131. a) Two people agree to play the following game: they alternately randomly draw marbles without replacement from an urn containing 4 white and 5 black marbles. Whoever removes the first white marble is the winner. What is the probability that the player who goes first will win the game?

{Hint: What draw sequences have the first player win? $\underbrace{(W)}_{\text{1 draw}},\underbrace{(B,B,W)}_{\text{3 draws}},\underbrace{(B,B,B,B,W)}_{\text{5 draws}}$*}*

b) Do the same problem except that now there are 5 white and 6 black marbles.

Do You Want To Go First Or Second?

132. a) Smith and Jones are both 50 percent marksmen. They decide to fight each other in a duel in which they exchange alternate shots until one of them is hit. If Smith shoots first, what is the probability that he wins the duel?

{Hint: Smith wins if any of the following exchanges occur:

Exchange	***Probability***
S	$\frac{1}{2}$
SJS	$\frac{1}{8}$
SJSJS	$\frac{1}{32}$
SJSJSJS	$\frac{1}{128}$
$\vdots$	$\vdots$

So the probability that Smith wins is $P = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$. *From this we can conclude that* $\frac{1}{4}P = \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots = P - \frac{1}{2}$*.}*

b) If Smith is a 40 percent marksman, and Jones is a 50 percent marksman. If Smith shoots first, what is the probability that he wins the duel?

Exchange	***Probability***
S	$\frac{2}{5}$
SJS	$\frac{6}{50}$
SJSJS	$\frac{18}{500}$
SJSJSJS	$\frac{54}{5000}$
$\vdots$	$\vdots$

{Hint: So the probability that Smith wins is $P = \frac{2}{5} + \frac{6}{50} + \frac{18}{500} + \frac{54}{5,000} + \cdots$. *From this we can conclude that* $\frac{3}{10}P = \frac{6}{50} + \frac{18}{500} + \frac{54}{5,000} + \cdots$*.}*

Illegally Blonde?

133. A woman who was shopping in Los Angeles had her purse stolen by a young, blonde female who was wearing a ponytail. The blonde female got into a yellow car that was driven by a black male who had a moustache and a beard. The police located a blonde female named Janet Collins who wore her hair in a ponytail and had a friend who was a black male with a moustache and beard who drove a yellow car. The police arrested the two subjects. Because there were no eyewitnesses and no real evidence, the prosecution used probability to make its case against the defendants. The probabilities listed below were presented by the prosecution for the known characteristics of the thieves.

Characteristic	**Probability**
Yellow car	$\frac{1}{10}$
Man with a moustache	$\frac{1}{4}$
Woman with a ponytail	$\frac{1}{10}$
Woman with blonde hair	$\frac{1}{3}$
Black man with beard	$\frac{1}{10}$
Interracial couple in car	$\frac{1}{1000}$

a) Assuming that all these characteristics are independent, what is the probability that a randomly selected couple has all these characteristics?

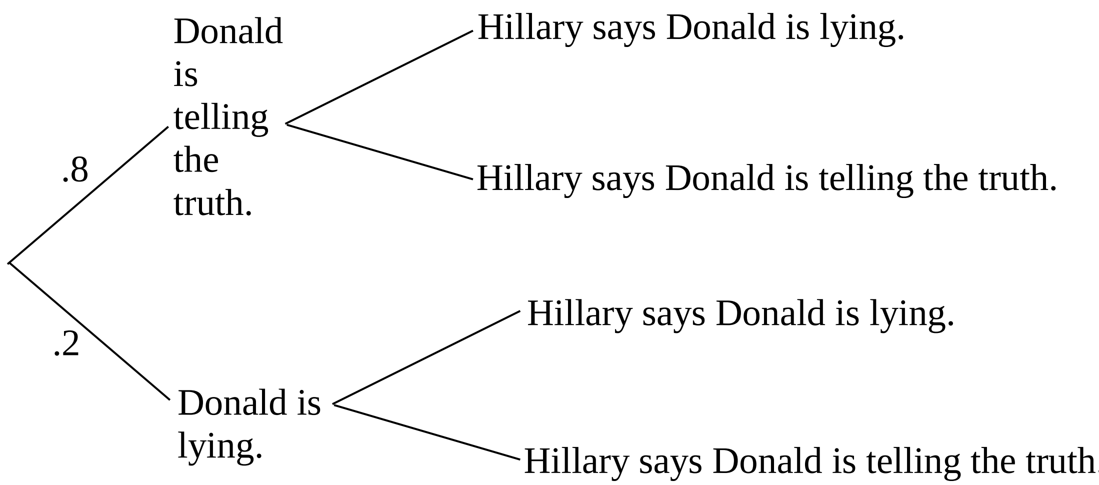
b) Would you convict the defendants based on this probability? Why?

c) Now let n represent the number of couples in the Los Angeles area who could have committed the crime. Let p represent the probability that a couple from this group has all 6 characteristics listed. Let X be the number of couples from this group who have all the 6 characteristics. Then $P(X=k)={}_nC_k p^k(1-p)^{n-k}$. Assuming that $n=1{,}000{,}000$ couples in the Los Angeles area and $p=\frac{1}{12{,}000{,}000}$, what is the probability that more than one of them has the 6 characteristics? Does this change your mind regarding the defendant's guilt?

d) Now let's look at this case from a different point of view. We will compute the probability that more than one couple has the characteristics described, given that at least one couple has the characteristics. $P(X>1\,|\,X\geq 1)=\dfrac{P(X>1 \text{ and } X\geq 1)}{P(X\geq 1)}=\dfrac{P(X>1)}{P(X\geq 1)}$. Compute this probability assuming that $n=1{,}000{,}000$ and $p=\frac{1}{12{,}000{,}000}$. Compute this again, but this time assume that $n=2{,}000{,}000$. Do you think that the couple should be convicted *beyond all reasonable doubt*? Why?

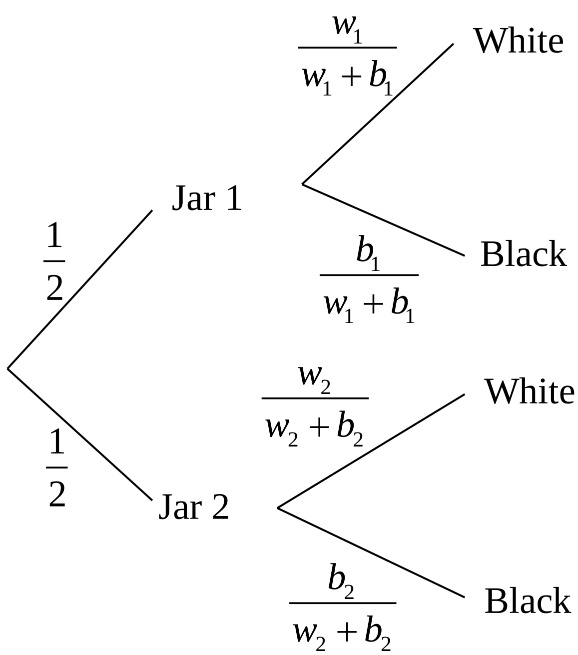
Liars, Damn Liars, And Politicians!

134. Donald tells the truth 80% of the time. Hillary tells the truth 75% of the time. What is the probability that Donald is telling the truth, given that Hillary says that Donald is lying? Assume that the events of Donald telling the truth and Hillary telling the truth are independent.



Don't Go Losing Your Head Over This One!

135. A convicted murderer is given 20 chips: 10 white and 10 black. All 20 chips are to be placed into two jars however the prisoner wishes. The executioner will randomly select one of the jars and then randomly select a chip from it. If the selected chip is white, the prisoner is set free; if the selected chip is black, the prisoner is executed. How should the chips be distributed to maximize the prisoner's chance of being set free?



$P(White)=\frac{1}{2}\left(\frac{w_1}{w_1+b_1}+\frac{w_2}{w_2+b_2}\right)$, and once w_1 and b_1 are selected, w_2 and b_2 are determined. Note: If $w_1=b_1=0$ or $w_1=b_1=10$, then $P(White)=\frac{1}{4}$.

b_1

w_1											
		(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)
	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)
	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)
	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)
	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)
w_1	(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)
	(6,0)	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	(6,8)	(6,9)	(6,10)
	(7,0)	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	(7,8)	(7,9)	(7,10)
	(8,0)	(8,1)	(8,2)	(8,3)	(8,4)	(8,5)	(8,6)	(8,7)	(8,8)	(8,9)	(8,10)
	(9,0)	(9,1)	(9,2)	(9,3)	(9,4)	(9,5)	(9,6)	(9,7)	(9,8)	(9,9)	(9,10)
	(10,0)	(10,1)	(10,2)	(10,3)	(10,4)	(10,5)	(10,6)	(10,7)	(10,8)	(10,9)	

See the link to the prisoner!

Let History Repeat Itself.

136. In a history test given to 100 students, 50 earned scores of 80 or more. Of the other students, 40 earned scores from 60 to 75(inclusive), and the other 10 earned scores from 10 to 40(inclusive).

a) What's the largest possible value of the mean?
b) What's the smallest possible value of the mean?
c) What's the largest possible value of the median?
d) What's the smallest possible value of the median?
e) What's the largest possible value of the mode?
f) What's the smallest possible value of the mode?
g) Can the mean be less than the median? Explain.
h) Can the mean be less than the mode? Explain.
i) Can the median be less than the mode? Explain.

Do The Math.

137. In a mathematics test given to 50 students, 25 earned scores of 90. Most of the other students scored 80, and the remaining students scored 30(at least one scored 30).

a) What's the largest possible value of the mean?
b) What's the smallest possible value of the mean?
c) What's the median?
d) What's the mode?
e) Can the mode be greater than the median? Explain.
f) Can the median be greater than the mean? Explain.
g) Can the mode be less than the mean? Explain.
h) Can the mean be greater than the median? Explain.

Two people, A and B, play a game in which a fair coin is repeatedly tossed. If it shows heads, then A wins \$1 from B. If it shows tails, then B wins \$1 from A. The two players start out with fixed amounts of money and continue playing the game until one of the players runs out of money. For example, suppose that A starts out with \$500 and B starts out with \$1000. Let $P(n)$ be the probability that A wins having reached \$$n$. Clearly, $P(0)=0$ since A will have run out of money. Let's find $P(1)$ using conditional probability: $\begin{aligned} P(1) &= P(1|0)P(0)+P(1|2)P(2) \\ &= \tfrac{1}{2}P(0)+\tfrac{1}{2}P(2) \end{aligned}$, so $P(2)=2P(1)-P(0)=2P(1)$. Using the same reasoning with $P(2)$, we get $P(2)=\tfrac{1}{2}P(1)+\tfrac{1}{2}P(3)$, so $P(3)=2P(2)-P(1)=4P(1)-P(1)=3P(1)$. In general, we get that $P(n)=nP(1)$ for $n\geq 0$. We still don't know the value of $P(1)$, but since $P(1500)=1$, we can solve for $P(1)$ from the equation $1500P(1)=1$ to get $P(1)=\frac{1}{1500}$. So the probability that A will win the game is $P(500)=500\cdot\frac{1}{1500}=\frac{1}{3}$. On average, how many plays of the game will occur until the game ends? Let $E(n)$ be the expected number of plays until A wins or runs out of money. Clearly, $E(0)=0$ since no more plays are required for A to run out of money. Let's find $E(1)$ using conditional expectation:

$$\begin{aligned}E(1)&=\left[1+E(0)\right]P(0)+\left[1+E(2)\right]P(2)\\&=1+\tfrac{1}{2}E(0)+\tfrac{1}{2}E(2)\end{aligned},$$

so $E(2)=2E(1)-2-E(0)=2E(1)-2$. Using the same reasoning with $E(2)$, we get $E(2)=1+\tfrac{1}{2}E(1)+\tfrac{1}{2}E(3)$, so $E(3)=2E(2)-2-E(1)=4E(1)-4-2-E(1)=3E(1)-6$. In general, we get that $E(n)=nE(1)-n(n-1)$ for $n\geq 0$. We still don't know the value of $E(1)$, but since $E(1500)=0$, we can solve for $E(1)$ from the equation $1500E(1)-1500\cdot 1499=0$ to get $E(1)=1499$. So $E(500)=500\cdot 1499-500\cdot 499=500000$.

Winner Takes All.

138. Two people, A and B, play a game in which a biased coin is repeatedly tossed. The coin shows heads with probability $\frac{2}{3}$, and tails with probability $\frac{1}{3}$. If it shows heads, then A wins \$1 from B. If it shows tails, then B wins \$1 from A. The two players start out with fixed amounts of money and continue playing the game until one of the players runs out of money.

a) If A starts out with \$5 and B with \$10, then find the probability that A wins. What's the expected number of plays until the game is finished?

$P(0)=0$, $P(1)=\frac{1}{3}P(0)+\frac{2}{3}P(2)\Rightarrow P(2)=\frac{3}{2}P(1)$.

$P(2)=\frac{1}{3}P(1)+\frac{2}{3}P(3)\Rightarrow P(3)=\frac{3}{2}P(2)-\frac{1}{2}P(1)=\frac{7}{4}P(1)$.

$P(3)=\frac{1}{3}P(2)+\frac{2}{3}P(4)\Rightarrow P(4)=\frac{3}{2}P(3)-\frac{1}{2}P(2)=\frac{15}{8}P(1)$. *In general,* $P(n)=\frac{2^n-1}{2^{n-1}}P(1)$, *and* $P(15)=1$, *so* $\frac{2^{15}-1}{2^{15-1}}P(1)=1\Rightarrow P(1)=\frac{16384}{32767}$. *This means that* $P(n)=\frac{2^n-1}{2^{n-1}}\cdot\frac{16384}{32767}$.

$E(0)=0$, $E(1)=\frac{1}{3}\left[1+E(0)\right]+\frac{2}{3}\left[1+E(2)\right]\Rightarrow E(2)=\frac{3}{2}E(1)-\frac{3}{2}$.

$E(2)=\frac{1}{3}\left[1+E(1)\right]+\frac{2}{3}\left[1+E(3)\right]\Rightarrow E(3)=\frac{7}{4}E(1)-\frac{15}{4}$.

$E(3)=\frac{1}{3}\left[1+E(2)\right]+\frac{2}{3}\left[1+E(4)\right]\Rightarrow E(4)=\frac{15}{8}E(1)-\frac{51}{8}$. *In general,*

$E(n)=\frac{2^n-1}{2^{n-1}}E(1)-\frac{2^{n-1}(3n-6)+3}{2^{n-1}}$, *and* $E(15)=0$, *so*

$\frac{2^{15}-1}{2^{14}}E(1)=\frac{2^{14}(45-6)+3}{2^{14}}\Rightarrow E(1)=\frac{2^{14}\cdot 39+3}{2^{15}-1}=\frac{638979}{32767}\approx 19.5$ *This means that*

$E(n)=\frac{2^n-1}{2^{n-1}}\cdot\frac{638979}{32767}-\frac{2^{n-1}(3n-6)+3}{2^{n-1}}$.

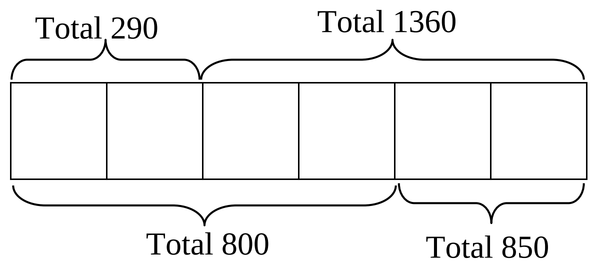
b) If A starts out with \$10 and B with \$20, then find the probability that A wins. What's the expected number of plays until the game is finished?

c) If A starts out with \$500 and B with \$1000, then find the probability that A wins. What's the expected number of plays until the game is finished?

See the link to Winner Takes All!

The Means Justify The Means.

139. The mean of six distinct numbers is 275. The mean of the four smallest numbers is 200. The mean of the four largest numbers is 340. Find the mean of the middle two numbers.



Oddly, It's As Easy As 1, 2, 3, 4, 5.

140. a) From the digits 1, 2, 3, 4, 5, how many four-digit numbers with distinct digits can be constructed?

b) Of these, how many are odd?

Is It Broken If It's Out Of Alphabetical Order?

141. Find the number of 5 letter permutations of the letters A, B, C, D,…, Z that are in alphabetical order.

Vote As Many Times As You Can.

142. In an election, each voter can distribute up to 5 votes among 6 candidates. For example, you could cast 3 votes for one candidate and 2 for another, or you could cast 1 vote for each of 4 candidates and not cast your fifth vote. In how many ways can you distribute your votes?

{Hint:

Candidate 1	Candidate 2	Candidate 3	Candidate 4	Candidate 5	Candidate 6	No one

See the hint for #5.}

Distinctly Odd, Or Oddly Distinct?

143. a) How many whole numbers between 1000 and 9999 have distinct digits?

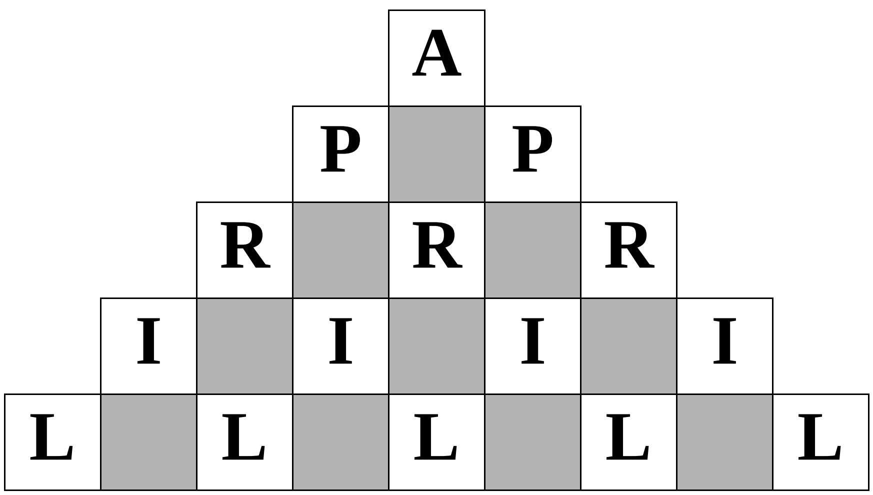
b) Of these, how many are odd numbers?

"No Orange For You!" Said The Fruit Nazi.

144. A father has 5 distinct oranges which he gives to his 8 sons so that each son either receives one orange or none. How many different ways can he do this?
{Hint: Use the formula for permutations with duplicates for 5 different oranges and 3 identical non-oranges, and use the position of the orange or non-orange as the son who gets that result.}

A Springtime Path.

145. Determine the number of different paths for spelling the word APRIL:



{Hint: The letters essentially form a tree diagram.}

Have A Seat!

146. In a classroom, there are 28 chairs. If 26 students are to be seated in the classroom, how many different ways can this be done?
{Hint: Rather than assign students to seats, assign seats to students.}

Well Jenny, It Can't Be 867-5309!

147. John is having trouble remembering his girlfriend Jenny's 7-digit phone number. He remembers that the first four digits consist of one 1, one 2, and two 3s. He also remembers that the fifth digit is either a 4 or 5. While he has no memory of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. If this is all the information he has, how many possible phone numbers are there?

{Hint:

How many	*for the first*	*4?*		*How many here?*	*How many*	*here?*

}

Don't Be A Sloth-Two Or Three Toed.

148. a) How many whole numbers less than 1,000 contain no digits of 3 but at least one digit of 2?

{Hint: How many have no 3's? How many have no 2's?}

b) How many whole numbers less than 1,000,000 contain no digits of 3 but at least one digit of 2?

Are You An Ace At 52-card Pickup?

149. A deck of cards is randomly dealt out.

a) What is the probability that the fourth card dealt is an ace?

b) What is the probability that the first ace occurs on the fourth card?

Transformers.

150. In studying the cause of power failures, the following data has been gathered:

5% are due to transformer damage
80% are due to line damage
1% involve both

Find the probability that a given power failure involves

a) line damage given that there is transformer damage

b) transformer damage given that there is line damage

c) transformer damage but not line damage

d) transformer damage given that there is no line damage

e) transformer damage or line damage

Only If We Stay In Different Hotels.

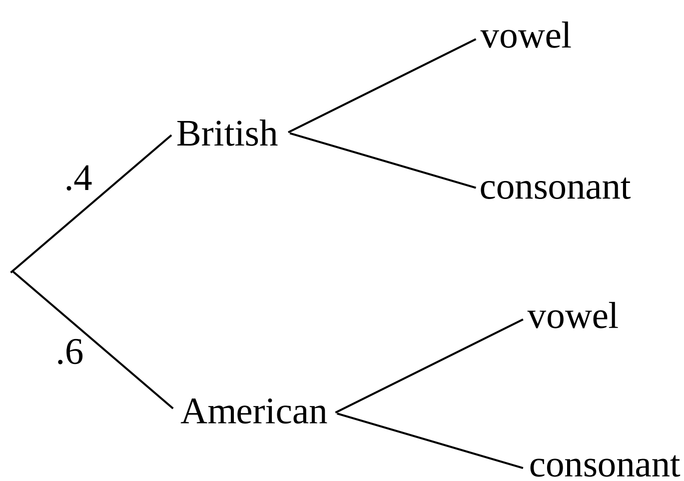
151. There are 5 hotels in a certain town. If 3 people randomly check into hotels in a day,

a) what is the probability that they each check into a different hotel?

b) what is the probability that at least two check into the same hotel?

Those Brits.

152. British and American spellings are *rigour* and *rigor*, respectively. A man staying at a Parisian hotel writes this word, and a letter taken at random from his spelling is found to be a vowel. If 40% of the English-speaking men at the hotel are British and 60% are American, what is the probability that the writer is British?



What Are The Odds Of The Odds Coming Up First?

153. The numbers one through seven are randomly drawn from a hat one at a time without replacement. What is the probability that all the odd numbers will be chosen first?

Three Heads Are Better Than Two.

154. Jacob flips five fair coins, exactly three of which land heads. What is the probability that the first three are heads?

{Hint: $P(\text{first three heads}|\text{exactly three heads}) = \dfrac{P(\text{first three heads and exactly three heads})}{P(\text{exactly three heads})}$ *or use reduced sample space.}*

Linus, Did You Lose Your Security Blanket?

155. Linus is lost on a line and starts to wander aimlessly. Each minute he walks 1 foot forward with probability $\frac{1}{2}$, stays where he is with probability $\frac{1}{3}$, and walks 1 foot backward with probability $\frac{1}{6}$. After one hour, what is the expected forward distance that Linus has walked?

{Hint: Let X_1 *be the forward distance that he walks after the first minute,* X_2 *the second minute,...,* X_{60} *the 60*th *minute. The forward distance he has walked after 1 hour is* $X = X_1 + X_2 + \cdots + X_{60}$.
You want $E(X) = E(X_1) + E(X_2) + \cdots + E(X_{60}) = 60E(X_1)$. *So all you have to do is find the expected forward progress after 1 minute and multiply it by 60.}*

Just Your Average Exam.

156. In a class of 50 students, the average score on an exam is 68. The best 10 exam scores are all 100. What's the average of the other 40 exams?

Mean, Median, Mode,…, Sum?

157. Five test scores have a mean of 91, a median of 93, and a mode of 95. The possible scores on the tests are from 0 to 100.

a) What is the sum of the lowest two test scores?

b) What are the possible values of the lowest two test scores?

The Median By Any Mode.

158. Fill-in the blanks in the list of numbers so that the mode of the list will be 8, and the median will be 13.

3,8,___,___,11,___,18,21,21,25

Making Money By Trimming The Hedges.

159. In an upcoming Super Bowl game, the two teams will be the Houston Texans and the Dallas Cowboys. Andy believes that Houston will win with probability $\frac{5}{8}$, while Betty believes that Dallas will win with probability $\frac{3}{4}$. You bet Andy that you will pay him $20 if Houston wins, and he will pay you $30 otherwise. You offer Betty $20 if Dallas wins, and she will pay you $30 otherwise. You can't lose here. Whether Houston or Dallas wins you will receive $30 from either Andy or Betty and pay out $20 to the other. You always receive $10 because you have hedged your bets in such a way that a loss of one is compensated by a bet on the opposite outcome. This is the basis of hedge fund financial investments. At the core, the hedge fund managers exploit differences in expectation about the same events to hedge against the risk of overall loss. Unfortunately, when this strategy is made known to the public it can seem unscrupulous. This was discovered by Goldman Sachs when it was revealed that they were encouraging clients to invest in options that Goldman Sachs were hedging against-in effect, betting that they would fail.

a) Determine Andy's expected payoff, and see why he will accept the bet.

b) Determine Betty's expected payoff, and see why she will accept the bet.

What's $882,000 Between Friends?

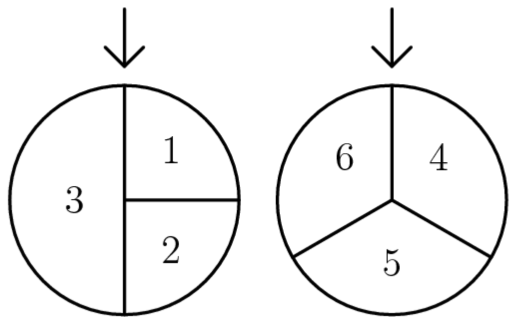
160. The annual incomes of 1,000 families were recorded. In error, the largest income was entered on the computer as $980,000, instead of the correct value of $98,000. What's the difference between the mean of the incorrect data and the mean of the actual data?

Anybody Can Add 'Em-Roll 'Em And Multiply.

161. If two fair dice are rolled, what's the probability that the product of the numbers showing is greater than 10?

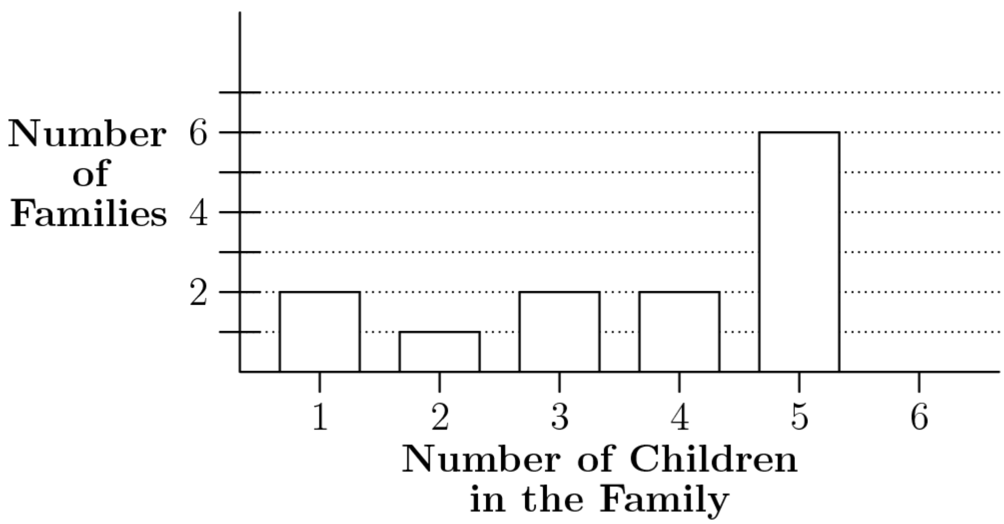
Spinning Wheels Got To Go Round.

162. The two wheels shown below are spun and the two resulting numbers are added. Find the probability that the sum is even.



Don't Let Your Children Play In The Median.

163. The graph shows the distribution of the number of children in the families of the students in Ms. Jordan's English class. What's the median number of children in the family for this distribution?



Sometimes Things Are The Same.

164. Diana and Apollo each roll a fair die obtaining a number from 1 to 6. What is the probability that Diana's number is larger than Apollo's number?

Carefully Eliminate.

165. What number should be removed from the list $\{1,2,3,4,5,6,7,8,9,10,11\}$ so that the mean of the remaining numbers is 6.1?

The Penultimate Pick.

166. A box contains five cards numbered 1, 2, 3, 4, and 5. Three cards are selected at random without replacement from the box. What's the probability that 4 is the largest value selected?