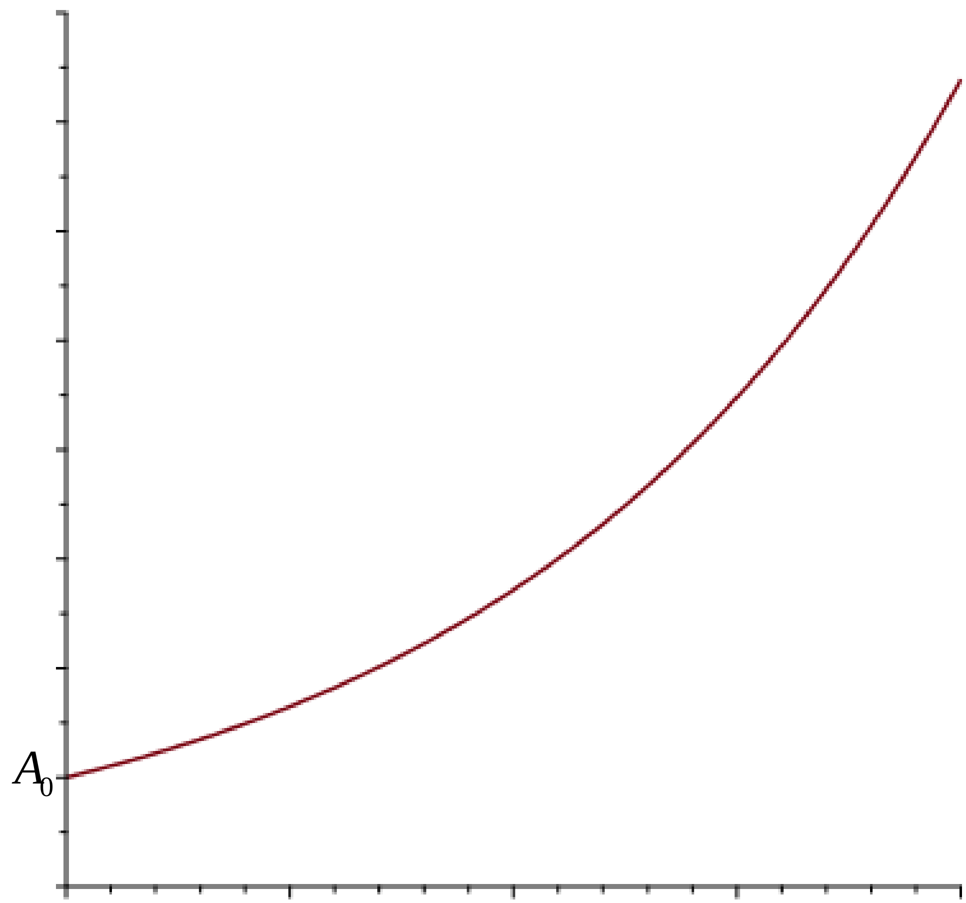
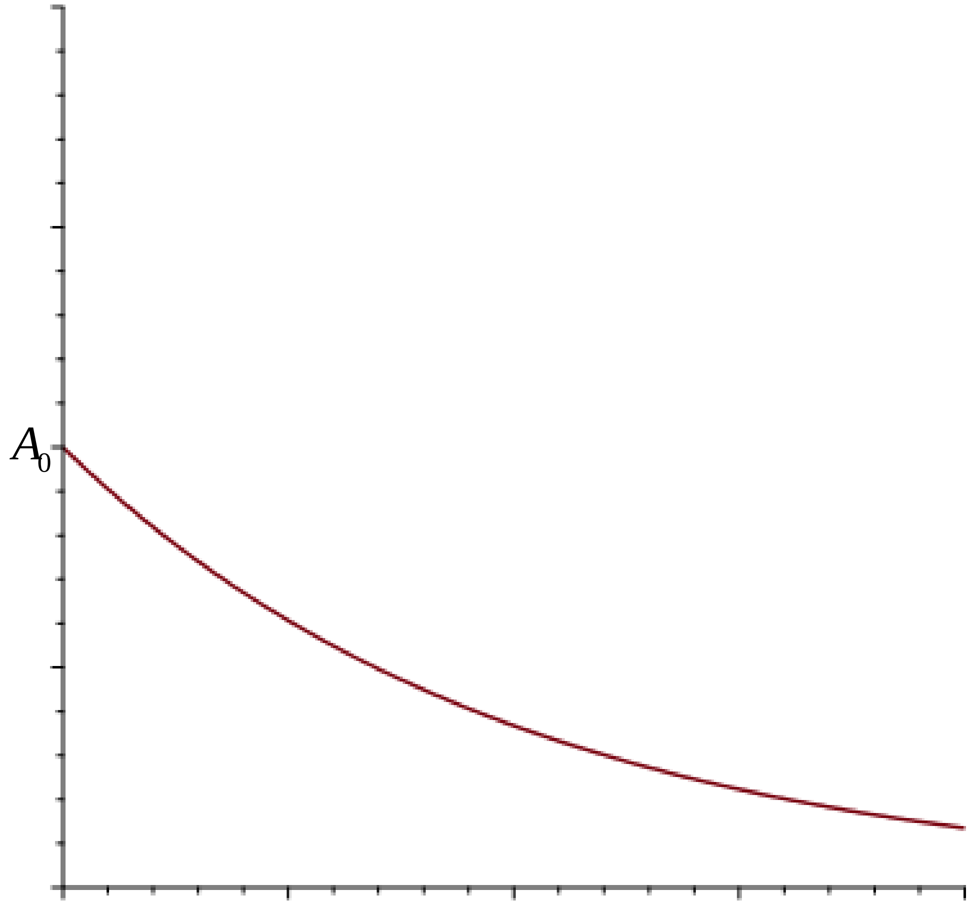
Exponential Growth and Decay Models:

$$A(t) = A_0 e^{kt}; t \geq 0, k \neq 0$$

For $k > 0$, the function models unlimited(exponential) growth, and *k* is called the growth rate.



For $k < 0$, the function models exponential decay, and *k* is called the decay rate.



Examples:

1. The number of bacteria in a culture is modeled by the exponential growth function $A(t) = 1000e^{.01t}$, where *t* is measured in hours.

a) What is the initial number of bacteria?

$$1000e^{.01 \cdot 0} = 1000 \cdot 1 = \boxed{1000}$$

b) What is the population after 4 hours?

$$A(4) = 1000e^{.04} = 1040.8107... \boxed{\approx 1041}$$

c) When will the number of bacteria reach 1700?

$$1000e^{.01t} = 1700 \Rightarrow e^{.01t} = 1.7$$

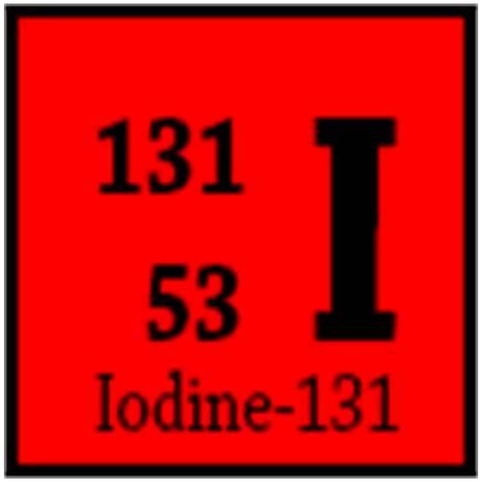
$$\Rightarrow .01t = ln(1.7) \Rightarrow t = \frac{ln(1.7)}{.01} = 53.0628... \boxed{\approx 53hrs}$$

d) When will the number of bacteria double?

$$1000e^{.01t} = 2000 \Rightarrow e^{.01t} = 2$$

$$\Rightarrow .01t = ln(2) \Rightarrow t = \frac{ln(2)}{.01} = 69.3147... \boxed{\approx 69hrs}$$

2. **The decay of Iodine-131 is modeled by the exponential decay function $A(t)=100e^{-.087t}$, where t is in days, and the amount of Iodine is in grams.**



a) What is the initial amount of Iodine?

$$100e^{-.087\cdot 0}=100\cdot 1=\boxed{100grams}$$

b) How much Iodine is left after 9 days?

$$A(9)=100e^{-.087(9)}=45.703...\approx\boxed{46g}$$

c) When will 70 grams of Iodine be left?

$$100e^{-.087t}=70\Rightarrow e^{-.087t}=.7\Rightarrow -.087t=ln(.7)$$

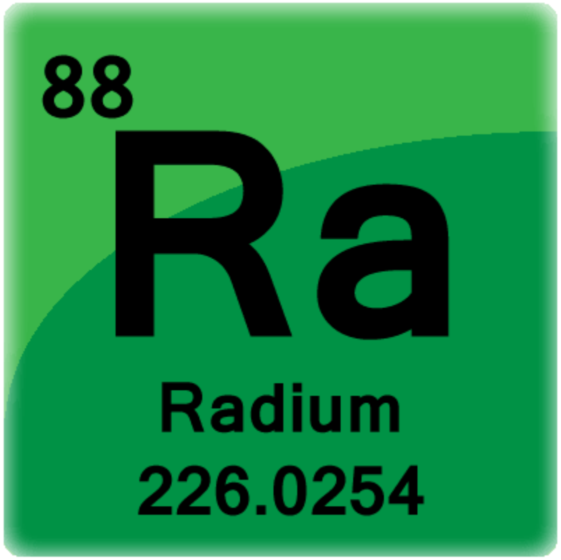
$$\Rightarrow t=\frac{ln(.7)}{-.087}=4.0997...\boxed{\approx 4days}$$

d) What is the half-life of iodine-131?

$$100e^{-.087t}=50\Rightarrow e^{-.087t}=.5\Rightarrow -.087t=ln(.5)$$

$$\Rightarrow t=\frac{ln(.5)}{-.087}=7.9672...\boxed{\approx 8days}$$

3. The half-life of Radium is 1690 years. If 10 grams is present now, how much will be present in 50 years?



$$A(t) = 10e^{kt} \Rightarrow 10e^{1690k} = 5 \Rightarrow e^{1690k} = .5$$

$$\Rightarrow 1690k = ln(.5) \Rightarrow k = \frac{ln(.5)}{1690}$$

$$\Rightarrow A(t) = 10e^{\frac{t\,ln(.5)}{1690}}$$

$$A(50) = 10e^{\frac{50\,ln(.5)}{1690}} = 9.797... \boxed{\approx 9.8g}$$

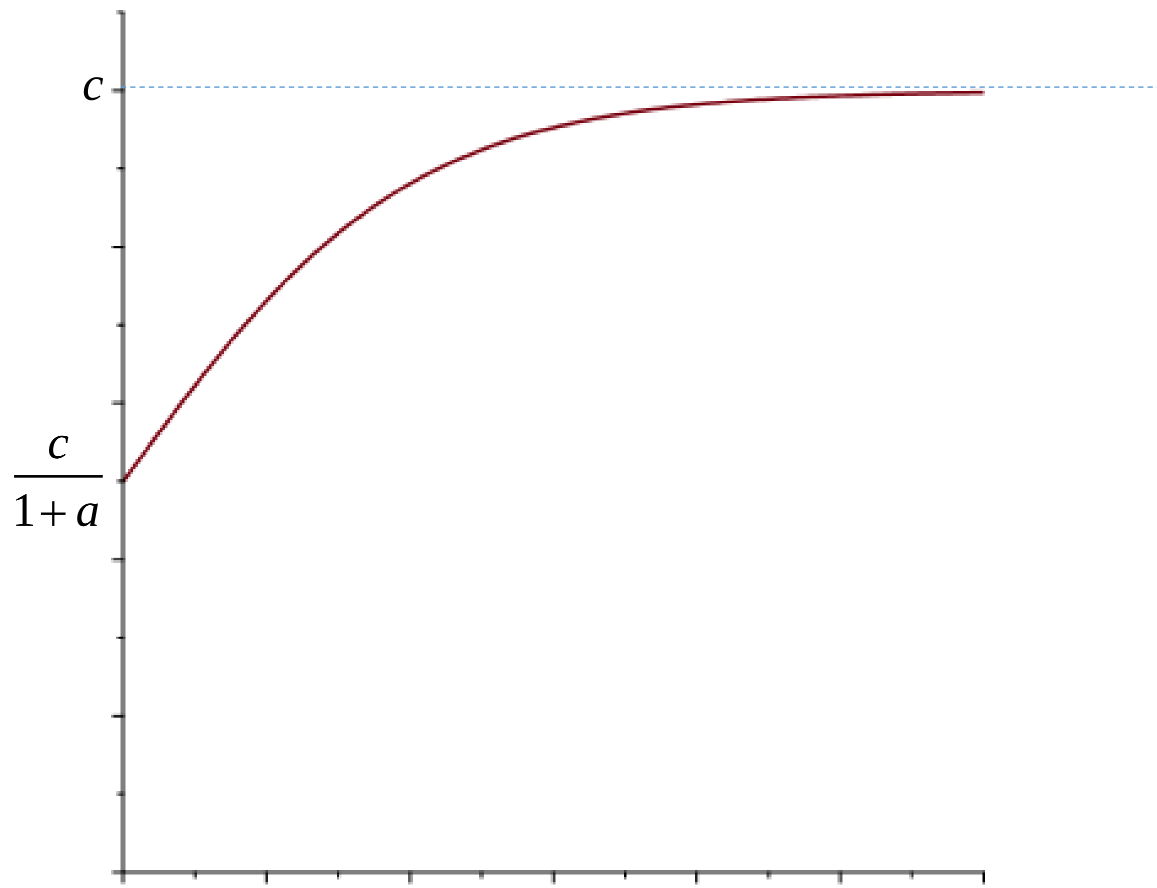
Logistic Growth and Decay Models:

$$P(t) = \frac{c}{1 + ae^{-bt}}; t \geq 0, a > 0, c > 0, b \neq 0$$

For $b > 0$, the function models limited(logistic) growth, and c is called the carrying capacity.

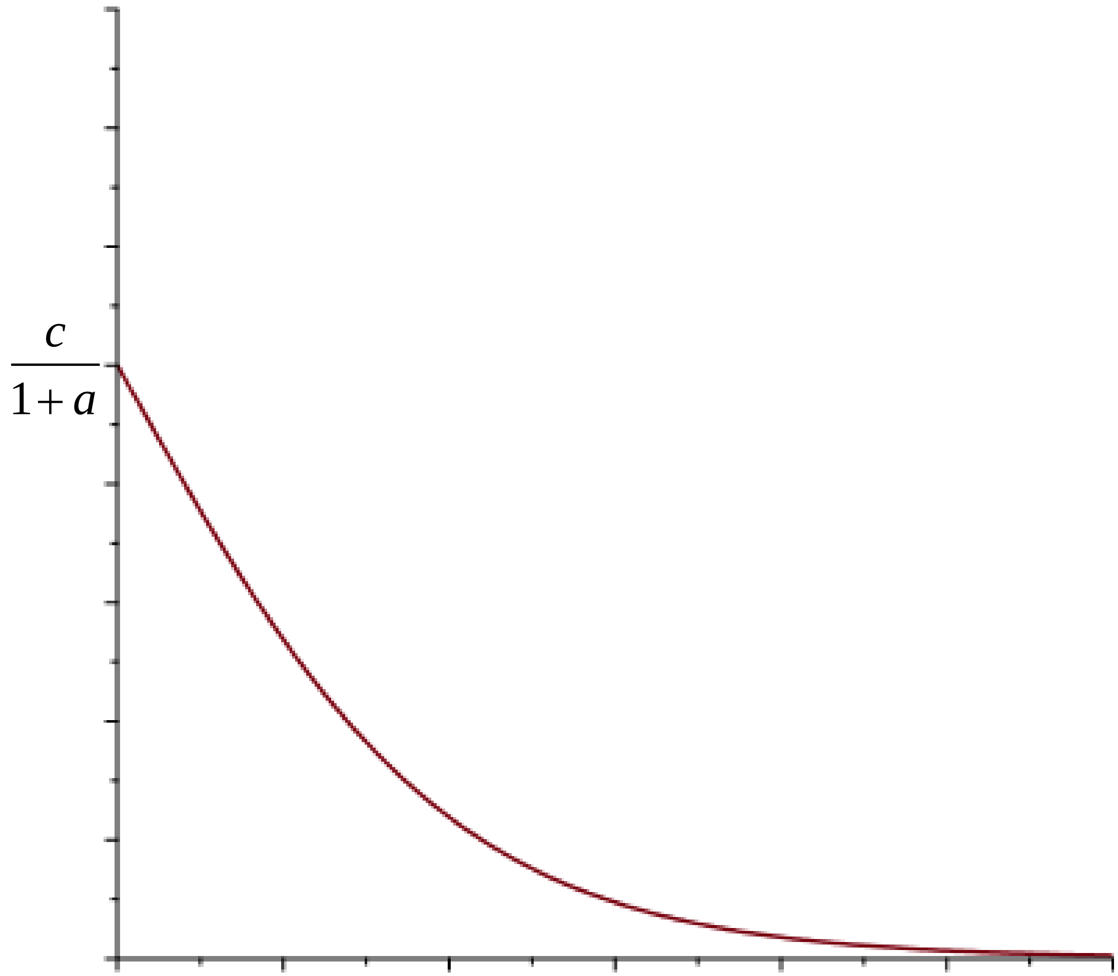
For t large, $\frac{c}{1 + ae^{-bt}} \approx \frac{c}{1} = c$

0



For $b < 0$, the function models logistic decay. Similar to exponential decay.

For t large, $\frac{c}{1+ae^{-bt}} \approx 0$



Example:

A model for the percentage of companies using Microsoft Word is the logistic growth function $P(t)=\dfrac{99.744}{1+3.01e^{-.799t}}$ **where *t* is the number of years since the end of 1984.**

a) What was the percentage of Word users at the end of 1984?

$$P(0)=\frac{99.744}{1+3.01e^{-.799(0)}}=\frac{99.744}{4.01}=24.8738\ldots\boxed{\approx 24.9\%}$$

b) What was the percentage of Word users at the end of 1990?

$$P(6)=\frac{99.744}{1+3.01e^{-.799(6)}}=97.3187\ldots\boxed{\approx 97.3\%}$$

c) When did the percentage of Word users reach 90%?

$$\frac{99.744}{1+3.01e^{-.799t}}=90 \Rightarrow 90\left(1+3.01e^{-.799t}\right)=99.744 \Rightarrow 1+3.01e^{-.799t}=\frac{99.744}{90}$$

$$\Rightarrow e^{-.799t}=\frac{\frac{99.744}{90}-1}{3.01} \Rightarrow t=\frac{ln\left(\frac{\frac{99.744}{90}-1}{3.01}\right)}{-.799}=4.1615\ldots\boxed{\approx 4.2\,yrs\text{ after the end of 1984}}$$

d) What is the carrying capacity percentage for Word users?

99.744%

Sequences:

A sequence is an ordered list of infinitely many numbers.

They can be represented by implying a pattern*(partial list)*, giving a direct formula, or giving a recursive formula.

Implying a pattern(partial list): ***{Determine the next two terms of each sequence.}***

$\{a_n\} = \{1,2,3,4,\ldots\}$

$5, 6$

$\{b_n\} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$

$\frac{1}{5}, \frac{1}{6}$

$\{c_n\} = \{1,-1,1,-1,\ldots\}$

$1, -1$

$\{d_n\} = \{1,-2,3,-4,5,-6,\ldots\}$

$7, -8$

$\{e_n\} = \{1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{5}, -\tfrac{1}{6}, \ldots\}$

$\frac{1}{7}, -\frac{1}{8}$

Direct Formula: **Unless stated otherwise, assume the starting subscript value is 1.**

{Determine the first five terms of each sequence.}

$a_n = n$

$1, 2, 3, 4, 5$

$b_n = \frac{1}{n}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

$c_n = (-1)^{n+1}$

$1, -1, 1, -1, 1$

$d_n = (-1)^{n+1} n$

$1, -2, 3, -4, 5$

$$e_n = \frac{(-1)^{n+1}}{n}$$

$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}$

<u>Recursive Formula:</u> ***{Determine the first five terms of each sequence.}***

$$a_1 = 1, a_{n+1} = 1 + a_n; n \geq 1$$

$$1, 2, 3, 4, 5$$

$$b_1 = 1, b_{n+1} = \frac{1}{1 + \frac{1}{b_n}}; n \geq 1$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

$$c_1 = 1, c_{n+1} = -c_n; n \geq 1$$

$$1, -1, 1, -1, 1$$

$$d_1 = 1, d_{n+1} = -d_n + (-1)^n; n \geq 1$$

$$1, -2, 3, -4, 5$$

$$e_1 = 1, e_{n+1} = \frac{1}{-\frac{1}{e_n} + (-1)^n}; n \geq 1$$

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}$$

$$f_1 = 1, f_2 = 1, f_{n+2} = f_{n+1} + f_n; n \geq 1$$

$$1, 1, 2, 3, 5$$

{Fibonacci}

Sigma or Summation Notation:

$a_1 + a_2 + a_3 + \cdots + a_n$ **can be abbreviated as** $\sum_{k=1}^{n} a_k$**.** **In other words,**

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n.$$

Examples:

1. Expand $\sum_{k=1}^{3} k^2$**.**

$1^2 + 2^2 + 3^2$

2. Expand $\sum_{k=1}^{4} (-1)^k$**.**

$(-1) + 1 + (-1) + 1$

3. Compress $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$**.**

$$\sum_{k=1}^{5} \frac{1}{k}$$

$\sum_{k=1}^{n} a_k$ **is considered to be a sum of a portion of the terms of the sequence** $\{a_1, a_2, a_3, \ldots\}$ **, and is sometimes referred to as a *finite series*.**

Properties of Finite Series: If $\{a_n\}$ and $\{b_n\}$ are sequences and c is any real number, then

1. $\sum_{k=1}^{n} (ca_k) = c \sum_{k=1}^{n} a_k$

Why? $\sum_{k=1}^{n} (ca_k) = (ca_1 + ca_2 + \cdots + ca_n) = c(a_1 + a_2 + \cdots + a_n) = c \sum_{k=1}^{n} a_k$

2. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Why? $\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + \cdots + (a_n + b_n) = (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)$

$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

3. $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

Why? $$\sum_{k=1}^{n}(a_k - b_k) = (a_1 - b_1) + \cdots + (a_n - b_n) = (a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots + b_n)$$

$$= \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

4. $$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k \quad \text{for } 1 \le j \le n-1$$

Why? $$\sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k = (a_1 + a_2 + \cdots + a_j + a_{j+1} + \cdots + a_n) - (a_1 + a_2 + \cdots + a_j)$$

$$= a_{j+1} + \cdots + a_n = \sum_{k=j+1}^{n} a_k$$

Special Formulas for Finite Series:

$$\sum_{k=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{(n \text{ terms})} = nc$$

Examples:

1. $\sum_{k=1}^{5} 2$

$5 \cdot 2 = \boxed{10}$

2. $\sum_{k=1}^{5,000} 3$

$5,000 \cdot 3 = \boxed{15,000}$

3. $\sum_{k=12}^{200} 2$

$$\left\{ \sum_{k=12}^{200} 2 = \sum_{k=1}^{200} 2 - \sum_{k=1}^{11} 2 \right\}$$

$400 - 22 = \boxed{378}$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$$ ***(the sum of the first n counting numbers)***

Let $S = 1 + 2 + 3 + \cdots + (n-1) + n$**. Then also,** $S = n + (n-1) + \cdots + 2 + 1$**.**

$$\begin{array}{rl} S = & 1 + 2 + 3 + \cdots + (n-1) + n \\ +S = & n + (n-1) + (n-2) + \cdots + 2 + 1 \\ \hline 2S = & \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} \end{array}$$

$$\Rightarrow 2S = n(n+1) \Rightarrow S = \frac{n(n+1)}{2}$$

So $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$**.**

Examples:

1. $\sum_{k=1}^{100} k$

$$\frac{100 \cdot 101}{2} = \boxed{5{,}050}$$

2. $\sum_{k=1}^{100} (k+2)$

$$= \sum_{k=1}^{100} k + \sum_{k=1}^{100} 2 = 5{,}050 + 200 = \boxed{5{,}250}$$

3. $\sum_{k=1}^{100} (2k)$

$$= 2\sum_{k=1}^{100} k = 2 \cdot 5{,}050 = \boxed{10{,}100}$$

4. $\sum_{k=11}^{100} k$ $\quad \left\{ \sum_{k=11}^{100} k = \sum_{k=1}^{100} k - \sum_{k=1}^{10} k \right\}$

$$= 5{,}050 - \frac{10 \cdot 11}{2} = 5{,}050 - 55 = \boxed{4{,}995}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

(the sum of the squares of the first n counting numbers)

$$\sum_{k=1}^{n} (k+1)^3 - \sum_{k=1}^{n} k^3 = \left[2^3 + 3^3 + \cdots + (n+1)^3\right] - \left[1^3 + 2^3 + \cdots + n^3\right] = (n+1)^3 - 1$$

And

$$\sum_{k=1}^{n} (k+1)^3 - \sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} \left[(k+1)^3 - k^3\right] = \sum_{k=1}^{n} \left(k^3 + 3k^2 + 3k + 1 - k^3\right) = \sum_{k=1}^{n} \left(3k^2 + 3k + 1\right)$$

$$= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n$$

So

$$3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n = (n+1)^3 - 1.$$

This means that $\sum_{k=1}^{n} k^2 = \dfrac{(n+1)^3 - 1 - \dfrac{3n(n+1)}{2} - n}{3} = \dfrac{n(n+1)(2n+1)}{6}$.

Examples:

1. $\sum_{k=1}^{12} k^2$

$\dfrac{12 \cdot 13 \cdot 25}{6} = \boxed{650}$

2. $\sum_{k=1}^{12} \left(k^2 - 4\right)$

$= \sum_{k=1}^{12} k^2 - \sum_{k=1}^{12} 4 = 650 - 48 = \boxed{602}$

3. $\sum_{k=1}^{12} \left(2k^2 - k + 1\right)$

$= 2\sum_{k=1}^{12} k^2 - \sum_{k=1}^{12} k + \sum_{k=1}^{12} 1 = 2 \cdot 650 - \dfrac{12 \cdot 13}{2} + 12 = 1300 - 78 + 12 = \boxed{1{,}234}$