The Method of Finite Differences:

Sometimes it's possible to find a nice formula for the terms of a sequence. Given a sequence, you can form a new sequence by subtracting each term from the term that follows it. The new sequence is called the sequence of first differences.

Example: For the sequence $\{a_n\} = \{2,5,8,11,14,17,20,\ldots\}$

a_n	**2**		**5**		**8**		**11**		**14**		**17**		**20**
1st differences		**3**		**3**		**3**		**3**		**3**		**3**	

If the sequence of first differences is a constant sequence, then the terms of the original sequence can be generated by a linear formula: $a_n = An + B$**. Find the formula for this example, assuming that the sequence of first differences remains constant.**

$$a_n = An + B$$

$(n=1)$ $A + B = 2$

$(n=2)$ $2A + B = 5$

Subtracting the first equation from the second leads to $A = 3$, and from the first equation, we get $B = -1$.

So the formula is $a_n = 3n - 1$.

The differences of the sequence of first differences can also be calculated to get the sequence of second differences.

Example: For the sequence $\{a_n\} = \{0,4,10,18,28,40,54,70,88,\ldots\}$

a_n	**0**		**4**		**10**		**18**		**28**		**40**		**54**		**70**		**88**
1st differences		**4**		**6**		**8**		**10**		**12**		**14**		**16**		**18**	
2nd differences			**2**		**2**		**2**		**2**		**2**		**2**		**2**		

If the sequence of second differences is a constant sequence, then the terms of the original sequence can be generated by a quadratic formula: $a_n = An^2 + Bn + C$**. Find the formula for this example, assuming that the sequence of second differences remains constant.**

$$a_n = An^2 + Bn + C$$

$(n=1)$ $A+B+C=0$

$(n=2)$ $4A+2B+C=4$

$(n=3)$ $9A+3B+C=10$

Subtracting the first equation from the second and third leads to $\begin{matrix} 3A+B=4 \\ 8A+2B=10 \end{matrix}$, and

dividing the second equation by 2 leads to $\begin{matrix} 3A+B=4 \\ 4A+B=5 \end{matrix}$. Now subtract the first equation

from the second to get $A=1$. So $3+B=4 \Rightarrow B=1$, and $1+1+C=0 \Rightarrow C=-2$.

So the formula is $a_n = n^2 + n - 2$.

Similar results hold for sequences with constant third differences, and fourth differences, and ….   See the link Finite Differences.

Let's find a formula for the sequence $\{a_n\} = \{0, 7, 26, 63, 124, 215, 342, 511, \ldots\}$**. See the link Finite Difference XL.**

n	1	2	3	4	5	6	7	8
sequence values	0	7	26	63	124	215	342	511
first differences		7	19	37	61	91	127	169
second differences			12	18	24	30	36	42
third differences				6	6	6	6	6

So the terms of the sequence can be generated by a cubic formula $a_n = An^3 + Bn^2 + Cn + D$**.**

$$a_n = An^3 + Bn^2 + Cn + D$$

$$(n=1) \quad A+B+C+D=0$$

$$(n=2) \quad 8A+4B+2C+D=7$$

$$(n=3) \quad 27A+9B+3C+D=26$$

$$(n=4) \quad 64A+16B+4C+D=63$$

Subtract the first equation from the others to get $\begin{array}{c} 7A+3B+C=7 \\ 26A+8B+2C=26 \\ 63A+15B+3C=63 \end{array}$. Now subtract twice the first from the second and three times the first from the third to get $\begin{array}{c} 12A+2B=12 \\ 42A+6B=42 \end{array}$. Divide the first by 2 and the second by 6 to get $\begin{array}{c} 6A+B=6 \\ 7A+B=7 \end{array}$. Now subtract the first from the second to get $A=1 \Rightarrow B=0 \Rightarrow C=0 \Rightarrow D=-1$.

So the formula is $a_n = n^3 - 1$.

Let's use Finite Differences to find a formula for the sum of the first *n* counting numbers:

a_n	1		$1+2$		$1+2+3$		$1+2+3+4$		$1+2+3+4+5$
1st differences		2		3		4		5	
2nd differences			**1**		**1**		**1**		

$$a_n = An^2 + Bn + C$$

$$\begin{aligned} A+B+C &= 1 \\ 4A+2B+C &= 3 \\ 9A+3B+C &= 6 \end{aligned}$$

$$\begin{aligned} 3A+B &= 2 \\ 4A+B &= \frac{5}{2} \end{aligned} \quad \Rightarrow A = \frac{1}{2}, B = \frac{1}{2}, C = 0$$

So the formula is $a_n = \frac{1}{2}n^2 + \frac{1}{2}n = \dfrac{n(n+1)}{2}$.

Let's use Finite Differences to find a formula for the sum of the squares of the first *n* counting numbers:

a_n	1^2		1^2+2^2		$1^2+2^2+3^2$		$1^2+2^2+3^2+4^2$		$1^2+2^2+3^2+4^2+5^2$
1st differences		2^2		3^2		4^2		5^2	
2nd differences			**5**		**7**		**9**		
3rd differences				**2**		**2**			

$$a_n = An^3 + Bn^2 + Cn + D$$

$$A + B + C + D = 1$$
$$8A + 4B + 2C + D = 5$$
$$27A + 9B + 3C + D = 14$$
$$64A + 16B + 4C + D = 30$$

$$7A + 3B + C = 4$$
$$26A + 8B + 2C = 13$$
$$63A + 15B + 3C = 29$$

$$12A + 2B = 5$$
$$42A + 6B = 17$$

$$A = \frac{1}{3} \Rightarrow B = \frac{1}{2} \Rightarrow C = \frac{1}{6} \Rightarrow D = 0$$

So the formula is $a_n = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n = \dfrac{2n^3 + 3n^2 + n}{6} = \dfrac{n(n+1)(2n+1)}{6}$.

Be careful! There are some assumptions in the Method of Finite Differences.

For example, for this sequence $\{1,2,3,\ldots\}$**, what's the fourth term?**

$$a_n = (n-1)(n-2)(n-3)+n$$

If this is the actual formula, then the fourth term is actually $(4-1)(4-2)(4-3)+4=\boxed{10}$.

Arithmetic Sequences:

It's a sequence of the form $\{a_1, a_1+d, a_1+2d, a_1+3d, \ldots\}$**, where consecutive terms differ by the same value, *d*, called the common difference.**

A formula for the terms of an arithmetic sequence is $a_n = a_1 + (n-1)d$**.**

Examples:

1. For the arithmetic sequence $\{2, 8, 14, 20, 26, \ldots\}$**,**

a) Find the common difference.

$$8 - 2 = \boxed{6}$$

b) Find a formula for the value of the n^{th} **term of the sequence.**

$$a_n = 2 + (n-1)6 \Rightarrow \boxed{a_n = 6n - 4}$$

c) Find the 20^{th} **term in the sequence.**

$$a_{20} = 6(20) - 4 = 120 - 4 = \boxed{116}$$

d) Is 2146 a number in this sequence?

$$6n - 4 = 2146 \Rightarrow 6n = 2150 \Rightarrow n = 358\tfrac{1}{3} \Rightarrow \boxed{No}$$

e) Is 1034 a number in this sequence?

$$6n - 4 = 1034 \Rightarrow 6n = 1038 \Rightarrow n = 173 \Rightarrow \boxed{Yes}$$

2. Find a formula for the n^{th} term of the arithmetic sequence with 4^{th} term of 3 and 20^{th} term of 35.

$a_n = a_1 + (n-1)d$ **, so** $a_1 + 3d = 3$

$$a_1 + 19d = 35$$

Subtracting the first equation from the second $\Rightarrow 16d = 32 \Rightarrow d = 2$

$\Rightarrow a_1 = -3 \Rightarrow a_n = -3 + (n-1)2 \Rightarrow \boxed{a_n = 2n-5}$

3. How many numbers are in the partial arithmetic sequence $\{8,5,2,-1,-4,\ldots,-295\}$?

$a_n = 8 + (n-1)(-3) = -3n + 11, -3n + 11 = -295 \Rightarrow -3n = -306 \Rightarrow n = \boxed{102}$

4. Find x so that $2x, 3x+2, 5x+3$ are consecutive terms in an arithmetic sequence.

$(3x+2) - 2x = d$ **and** $(5x+3) - (3x+2) = d$

$$(3x+2) - 2x = (5x+3) - (3x+2) \Rightarrow x + 2 = 2x + 1$$

$$\Rightarrow x = \boxed{1}$$

<u>The sum of the first n terms of an arithmetic sequence:</u>

$$
\begin{aligned}
S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\
&= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + \left[a_1 + (n-1)d\right] \\
&= na_1 + \left[1 + 2 + 3 + \cdots + (n-1)\right]d \\
&= na_1 + \left[\frac{n(n-1)}{2}\right]d \\
&= n\left[a_1 + \frac{(n-1)d}{2}\right] \\
&= n\left[\frac{a_1 + \left[a_1 + (n-1)d\right]}{2}\right] \\
&= n\left[\frac{a_1 + a_n}{2}\right]
\end{aligned}
$$

Yes, it's just *n* times the average of the first and last terms!

Find the following sums of partial arithmetic sequences:

1. $1+3+5+\cdots+(2n-1)$

$$n\left[\frac{1+(2n-1)}{2}\right]=\boxed{n^2}$$

2. $2+5+8+\cdots+41$

$$a_n=2+(n-1)3=3n-1$$

$$3n-1=41\Rightarrow 3n=42\Rightarrow n=14$$

$$14\left(\frac{2+41}{2}\right)=\boxed{301}$$

3. $7+1-5-11-\cdots-299$

$$a_n=7+(n-1)(-6)=-6n+13$$

$$-6n+13=-299\Rightarrow -6n=-312\Rightarrow n=52$$

$$52\left(\frac{7+(-299)}{2}\right)=\boxed{-7592}$$

4. $\sum_{n=1}^{90}(3-2n)$

$$90\left[\frac{1+(-177)}{2}\right]=\boxed{-7920}$$

5. The sum of the first 46 terms of the sequence $\{2,-1,-4,-7,\ldots\}$

$$a_n=2+(n-1)(-3)=-3n+5$$

$$46\left(\frac{2+(-133)}{2}\right)=\boxed{-3013}$$

How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is -4 to get a sum of 702?

$$n\left[\frac{\underbrace{78}_{a_1}+\underbrace{78+(n-1)(-4)}_{a_n}}{2}\right]=702$$

$$n\left[\frac{-4n+160}{2}\right]=702$$

$$n(-2n+80)=702$$

$$-2n^2+80n=702$$

$$2n^2-80n+702=0$$

$$n^2-40n+351=0$$

1	**351**
3	**117**
9	**39**
13	**27**

Factor or quadratic formula!

$$n^2-40n+351=0 \Rightarrow (n-13)(n-27)=0 \Rightarrow n=\boxed{13,27}$$