Geometric Sequences:

It's a sequence of the form $\{a_1, a_1 r, a_1 r^2, a_1 r^3, \ldots\}$**, where each term beyond the first is the factor,** ***r*****, times the previous term.** ***r*** **is called the common ratio, and** $a_1 \neq 0$**.**

A formula for the terms of a geometric sequence is $a_n = a_1 r^{n-1}$ **.**

Examples:

1. For the geometric sequence $\left\{1, \frac{2}{5}, \frac{4}{25}, \frac{8}{125}, \frac{16}{625}, \ldots\right\}$**,**

a) Find the common ratio. $\frac{\frac{2}{5}}{1} = \boxed{\frac{2}{5}}$

b) Find a formula for the value of the ***n*****th term of the sequence.**

$$a_n = 1 \cdot \left(\frac{2}{5}\right)^{n-1} \Rightarrow \boxed{a_n = \left(\frac{2}{5}\right)^{n-1}}$$

c) Find the 8th term in the sequence.

$$a_8 = \left(\frac{2}{5}\right)^7 = \boxed{\frac{128}{78{,}125}}$$

d) Is $\frac{1{,}024}{9{,}765{,}625}$ **a number in this sequence?**

$$\left(\frac{2}{5}\right)^{n-1} = \frac{1{,}024}{9{,}765{,}625} \Rightarrow n = 11 \Rightarrow \boxed{Yes}$$

e) Is $\frac{4{,}096}{48{,}828{,}125}$ a number in this sequence?

$$\left(\frac{2}{5}\right)^{n-1} = \frac{4{,}096}{48{,}828{,}125} \Rightarrow n = 11.2435\ldots \Rightarrow \boxed{No}$$

2. Find a formula for the n^{th} term of the geometric sequence of real numbers with 3rd term of 3 and 6th term of 24.

? , ? , 3 , ? , ? , 24

$a_1 r^2 = 3, a_1 r^5 = 24$

Dividing the second equation by the first $\Rightarrow r^3 = 8 \Rightarrow r = 2 \Rightarrow \boxed{a_n = \left(\frac{3}{4}\right) 2^{n-1}}$

3. Find *x* so that $x-1, x, x+2$ are consecutive terms in a geometric sequence.

$$\frac{x}{x-1} = \frac{x+2}{x} \Rightarrow x^2 + x - 2 = x^2 \Rightarrow x - 2 = 0 \Rightarrow x = \boxed{2}$$

The sum of the first n terms of a geometric sequence:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$$

So

If $r = 1$**, then** $S_n = a_1 + a_1 + a_1 + \cdots + a_1 = na_1$**.**

Otherwise

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} \\ -rS_n &= \qquad a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + a_1 r^n \\ \hline (1-r)S_n &= a_1 - a_1 r^n \end{aligned}$$

$$S_n = \frac{a_1 - a_1 r^n}{1-r} = a_1 \cdot \frac{1-r^n}{1-r}$$

So $S_n = \begin{cases} na_1 & ;r=1 \\ a_1 \cdot \dfrac{1-r^n}{1-r} & ;r \neq 1 \end{cases}$.

Find the following sums:

1. $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\cdots+\frac{2^{n-1}}{4}$

What's *r*? 2 **How many terms?** n **What's a_1?** $\frac{1}{4}$

$$=\frac{1}{4}\cdot 2^{1-1}+\frac{1}{4}\cdot 2^{2-1}+\frac{1}{4}\cdot 2^{3-1}+\cdots+\frac{1}{4}\cdot 2^{n-1}$$

$$S_n=\underbrace{\frac{1}{4}}_{a_1}\cdot\frac{1-2^n}{1-2}=\frac{2^n-1}{4}$$

2. $\frac{3}{9}+\frac{3^2}{9}+\frac{3^3}{9}+\cdots+\frac{3^{n+1}}{9}$

What's *r*? 3 **How many terms?** $n+1$ **What's a_1?** $\frac{3}{9}$

$$=\frac{3}{9}\cdot 3^0+\frac{3}{9}\cdot 3^1+\cdots+\frac{3}{9}\cdot 3^n$$

$$S_{n+1}=\underbrace{\frac{3}{9}}_{a_1}\cdot\frac{1-3^{n+1}}{1-3}=\frac{\left(3^{n+1}-1\right)}{6}$$

3. $\sum_{k=1}^{n-1}\left(\frac{2}{3}\right)^k$

What's r? $\frac{2}{3}$

How many terms? $n-1$

What's a_1? $\frac{2}{3}$

$$S_{n-1} = \underbrace{\frac{2}{3}}_{a_1}\cdot\frac{1-\left(\frac{2}{3}\right)^{n-1}}{1-\frac{2}{3}} = 2\left[1-\left(\frac{2}{3}\right)^{n-1}\right]$$

4. $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\cdots+\frac{2^{14}}{4}$

What's r? 2

How many terms? 15

What's a_1? $\frac{1}{4}$

$$S_{15} = \underbrace{\frac{1}{4}}_{a_1}\cdot\frac{1-2^{15}}{1-2} = \frac{2^{15}-1}{4} = \boxed{\frac{32{,}767}{4}}$$

5. $\sum_{n=0}^{8}\left(-\frac{2}{3}\right)^{n}$

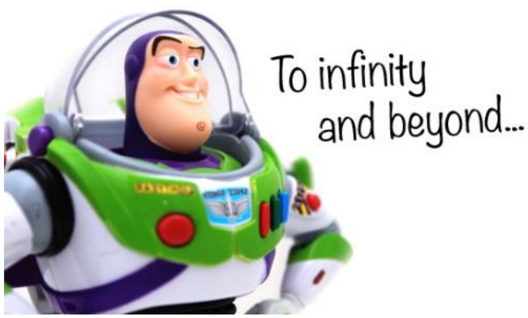
What's r? $-\frac{2}{3}$

How many terms? 9

What's a_1? 1

$$S_9 = \underset{a_1}{1}\cdot\frac{1-\left(-\frac{2}{3}\right)^9}{1-\left(-\frac{2}{3}\right)} = \frac{3}{5}\left[1+\left(\frac{2}{3}\right)^9\right] = \frac{3}{5}\left[\frac{19{,}683}{19{,}683}+\frac{512}{19{,}683}\right] = \boxed{\frac{4{,}039}{6{,}561}}$$

What if you try to add all the infinitely many terms of a geometric sequence?



Let's see what happens to the sum of the first *n* terms, called the *n*th partial sum, as the number of included terms gets larger. If it settles on a value we'll say that the partial sums converge to that value. Otherwise, we'll say that the partial sums diverge.

If $r=1$, $S_n = na_1$, and as *n* gets larger and larger, the *n*th partial sum, S_n, will head toward ∞, if $a_1 > 0$, or toward $-\infty$, if $a_1 < 0$. In this case, the partial sums diverge.

For $a_1 = 5$, let's look at the partial sums:

$S_1 = 5$	$S_2 = 5+5$	$S_3 = 5+5+5$	$S_4 = 5+5+5+5$	$S_5 = 5+5+5+5+5$	$\ldots$
5	**10**	**15**	**20**	**25**	$\ldots$

For $a_1 = -10$, let's look at the partial sums:

$S_1 = -10$	$S_2 = -10 + -10$	$S_3 = -10 + -10 + -10$	$S_4 = -10 + -10 + -10 + -10$	$\cdots$
-10	-20	-30	-40	$\cdots$

If $r = -1$, $S_1 = a_1$, $S_2 = a_1 - a_1 = 0$, $S_3 = a_1 - a_1 + a_1 = a_1$,…, so the partial sums will continue to bounce between a_1 and zero as *n* continues to increase. In this case, the partial sums diverge.

For $a_1 = 1$, let's look at the partial sums:

$S_1 = 1$	$S_2 = 1 + -1$	$S_3 = 1 + -1 + 1$	$S_4 = 1 + -1 + 1 + -1$	$S_5 = 1 + -1 + 1 + -1 + 1$
1	**0**	**1**	**0**	**1**

If $r=0$**,** $S_n = a_1 + 0 + 0 + \cdots + 0 = a_1$**, so the partial sums always have the value** a_1 **as *n* continues to increase. In this case, we say that the partial sums converge to** a_1**.**

For $a_1 = -2$**, let's look at the partial sums:**

$S_1 = -2$	$S_2 = -2+0$	$S_3 = -2+0+0$	$S_4 = -2+0+0+0$	$S_5 = -2+0+0+0+0$
-2	-2	-2	-2	-2

Since, for $r \neq 1$ $S_n = a_1 \cdot \dfrac{1-r^n}{1-r}$**, the only chance for it to settle is if** r^n **settles. This only happens when** $-1 < r < 1$**, in which case,** r^n **settles on zero.**

So $a_1 + a_1 r + a_1 r^2 + \cdots = \sum_{n=1}^{\infty} a_1 r^{n-1}$ **converges to** $a_1 \cdot \dfrac{1}{1-r}$ **if** $-1 < r < 1$**, and diverges otherwise. The attempt at adding up all the terms of a geometric sequence,** $a_1 + a_1 r + a_1 r^2 + \cdots = \sum_{n=1}^{\infty} a_1 r^{n-1}$**, is called an infinite geometric series, or just a geometric series.**

In short, a geometric series converges to $\dfrac{\text{first term}}{1-r}$ **if** $|r|<1$**, and diverges if** $|r|\geq 1$**.**

<u>Examples:</u>

Determine if the following geometric series are convergent or divergent. If convergent, tell what they converge to.

1. $2+\dfrac{4}{3}+\dfrac{8}{9}+\cdots$

$|r|=\left|\dfrac{2}{3}\right|=\dfrac{2}{3}<1 \Rightarrow$ converges to $\dfrac{2}{1-\frac{2}{3}}=6$

Here are the first 14 partial sums.

S_1	S_2	S_3	S_4	S_5	S_6	S_7
2	$\dfrac{10}{3}=3.\overline{3}$	$\dfrac{38}{9}=4.\overline{2}$	$\dfrac{130}{27}=4.\overline{814}$	$\dfrac{422}{81}\approx 5.21$	$\dfrac{1330}{243}\approx 5.47$	$\dfrac{4118}{729}\approx 5.65$
S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}
≈ 5.77	≈ 5.84	≈ 5.90	≈ 5.93	≈ 5.95	≈ 5.97	≈ 5.98

2. $8+4+2+\cdots$

$|r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1 \Rightarrow$ converges to $\frac{8}{1-\frac{1}{2}} = 16$

Here are the first 16 partial sums.

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
8	12	14	15	$\frac{31}{2} = 15.5$	$\frac{63}{4} = 15.75$	$\frac{127}{8} = 15.875$	$\frac{255}{16} = 15.9375$
S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
≈ 15.969	≈ 15.984	≈ 15.992	≈ 15.996	≈ 15.998	≈ 15.999	≈ 16.000	≈ 16.000

3. $2-\frac{1}{2}+\frac{1}{8}-\frac{1}{32}+\cdots$

$|r|=\left|-\frac{1}{4}\right|=\frac{1}{4}<1 \Rightarrow$ converges to $\frac{2}{1-\left(-\frac{1}{4}\right)}=\frac{8}{5}=1.6$

Here are the first 12 partial sums.

S_1	S_2	S_3	S_4	S_5	S_6
2	$\frac{3}{2}=1.5$	$\frac{13}{8}=1.625$	$\frac{51}{32}=1.59375$	$\frac{205}{128}=1.6015625$	$\frac{819}{512}=1.599609375$
S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
≈ 1.6000061	≈ 1.5999985	≈ 1.6000004	≈ 1.5999999	≈ 1.6000000	≈ 1.6000000

4. $\sum_{k=1}^{\infty} 3 \cdot \left(\frac{3}{2}\right)^{k-1}$

$|r| = \left|\frac{3}{2}\right| = \frac{3}{2} \geq 1 \Rightarrow \text{diverges}$

Here are the first 12 partial sums.

S_1	S_2	S_3	S_4	S_5	S_6
3	$\frac{15}{2} = 7.5$	$\frac{57}{4} = 14.25$	$\frac{195}{8} = 24.375$	$\frac{633}{16} = 39.5625$	$\frac{1995}{32} = 62.34375$
S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
≈ 96.52	≈ 147.77	≈ 224.66	≈ 339.99	≈ 512.99	≈ 772.48

5. $\sum_{n=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^n$

$$|r| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1 \Rightarrow \text{converges to } \frac{3}{1-\frac{2}{3}} = 9$$

Here are the first 14 partial sums.

S_1	S_2	S_3	S_4	S_5	S_6	S_7
3	5	$\frac{19}{3} = 6.\overline{3}$	$\frac{65}{9} = 7.\overline{2}$	$\frac{211}{27} = 7.\overline{814}$	$\frac{665}{81} \approx 8.21$	$\frac{2059}{243} \approx 8.47$
S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}
≈ 8.65	≈ 8.77	≈ 8.84	≈ 8.90	≈ 8.93	≈ 8.95	≈ 8.97

6. $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$

$$|r|=\left|-\frac{1}{2}\right|=\frac{1}{2}<1\Rightarrow \text{converges to } \frac{1}{1-\left(-\frac{1}{2}\right)}=\frac{2}{3}=.\overline{6}$$

Here are the first 16 partial sums.

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
1	$\frac{1}{2}=.5$	$\frac{3}{4}=.75$	$\frac{5}{8}=.625$	$\frac{11}{16}=.6875$	$\frac{21}{32}=.65625$	$\frac{43}{64}=.671875$	$\frac{85}{128}=.6640625$
S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
$\approx .66797$	$\approx .66602$	$\approx .66699$	$\approx .66650$	$\approx .66675$	$\approx .66663$	$\approx .66669$	$\approx .66666$

<u>Other Infinite Series:</u>

In general, all infinite series, $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$**, can be analyzed by looking at their related sequence of partial sums.**

Examples:

1. $\sum_{n=1}^{\infty} n$

It's sequence of partial sums is given by $S_n = 1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1)$**. Clearly, as** $n \to \infty$**, the values of the partial sums go to** ∞**, so this series is divergent.**

2. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

It's sequence of partial sums is given by $S_n=\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\tfrac{1}{n}-\tfrac{1}{n+1}\right)$**.**

$$=1-\tfrac{1}{n+1}$$

As $n\to\infty$**, the values of the partial sums go to 1, so this series converges to 1.**

3. $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$

It's sequence of partial sums is given by

$$\begin{aligned}S_n &= \ln\tfrac{1}{2}+\ln\tfrac{2}{3}+\ln\tfrac{3}{4}+\cdots+\ln\tfrac{n}{n+1} \\ &=(\ln 1-\ln 2)+(\ln 2-\ln 3)+(\ln 3-\ln 4)+\cdots+\left[\ln n-\ln(n+1)\right] \\ &=\ln 1-\ln(n+1) \\ &=-\ln(n+1)\end{aligned}$$

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As $n\to\infty$**, the values of the partial sums go to** $-\infty$**, so this series diverges.**