Proving Things Using the Principle of Mathematical Induction:

Mathematical Induction is used to prove statements about the natural/counting numbers.

It consists of two steps.

The first step*(base step)* is to show that the statement is true for the first natural number, 1.

The second step*(induction step)* is to show that if the statement is true for a natural number greater than or equal to 1, then it must also be true for the next natural number.

The reason why this combination proves the statement for all natural numbers is the following: Choose any natural number, *j*. The statement must be true for *j*, because it's true for 1 and therefore 2, and therefore 3,….., and therefore *j*.

Examples:

1. Prove that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ **for all natural numbers, *n*.**

<u>Base Step:</u> Show that it's true for $n=1$**.**

Left side	**Right side**
1	$\frac{1\cdot(1+1)}{2}=\frac{1\cdot 2}{2}=\boxed{1}$

Induction Step: **Show that *if* it's true for $n=k$, then it must also be true for $n=k+1$.**

Assuming that it's true for $n=k$, leads to

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}.$$

Goal: $1+2+3+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}$.

How do you turn the current left side into the desired left side of $1+2+3+\cdots+k+(k+1)$? Add $(k+1)$, so add it to both sides.

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

Conclusion:

$1+2+3+\cdots+n=\frac{n(n+1)}{2}$ for all natural numbers, n, by Mathematical Induction.

2. Prove that $1+2+2^2+\cdots+2^{n-1}=2^n-1$ for all natural numbers, *n*.

<u>Base Step:</u> Show that it's true for $n=1$.

Left side	**Right side**
$2^{1-1}=\boxed{1}$	$2^1-1=2-1=\boxed{1}$

<u>Induction Step:</u> Show that <u>if</u> it's true for $n = k$, then it must also be true for

$n = k + 1$**.**

<u>Assuming</u> that it's true for $n = k$, leads to

$1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$**.**

<u>Goal:</u> $1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^k = 2^{k+1} - 1$**.**

How do you turn the current left side into the desired left side of

$1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^k$**?** Add 2^k, so add it to both sides.

$$1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^k = 2^k - 1 + 2^k$$

$$= 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

<u>Conclusion:</u>

$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$ for all natural numbers, *n*, by Mathematical Induction.

3. Prove that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ **for all natural numbers, *n*.**

<u>Base Step:</u> Show that it's true for $n=1$**.**

Left side	Right side
$\frac{1}{1\cdot 2}=\boxed{\frac{1}{2}}$	$\frac{1}{1+1}=\boxed{\frac{1}{2}}$

<u>Induction Step:</u> **Show that <u>if</u> it's true for** $n = k$**, then it must also be true for** $n = k+1$**.**

<u>Assuming</u> that it's true for $n = k$**, leads to**

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}.$$

<u>Goal:</u> $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$**.**

How do you turn the current left side into the desired left side of

$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$**?** Add $\frac{1}{(k+1)(k+2)}$

, so add it to both sides.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

<u>Conclusion:</u> $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for all natural numbers, *n*, by Mathematical Induction.

4. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ **for all natural numbers, *n*.**

<u>Base Step:</u> Show that it's true for $n = 1$**.**

Left side	Right side
$1^3 = \boxed{1}$	$\dfrac{1^2(1+1)^2}{4} = \dfrac{1 \cdot 4}{4} = \boxed{1}$

<u>Induction Step:</u> **Show that <u>if</u> it's true for** $n = k$**, then it must also be true for** $n = k + 1$**.**

<u>Assuming</u> that it's true for $n = k$**, leads to**

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2 (k+1)^2}{4}.$$

<u>Goal:</u> $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2 (k+2)^2}{4}$.

How do you turn the current left side into the desired left side of $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$**?** Add $(k+1)^3$, so add it to both sides.

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2 (k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2}{4}\left(k^2 + 4k + 4\right) = \frac{(k+1)^2 (k+2)^2}{4}$$

<u>Conclusion:</u>

$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2 (n+1)^2}{4}$ for all natural numbers, n, by Mathematical Induction.

5. Prove that $2^n > n$ for all natural numbers, *n*.

<u>Base Step:</u> Show that it's true for $n = 1$.

Left side	**Right side**
$2^1 = \boxed{2}$	**1**

<u>Induction Step:</u> Show that <u>if</u> it's true for $n = k$, then it must also be true for

$n = k + 1$.

<u>Assuming</u> that it's true for $n = k$, leads to

$2^k > k$.

<u>Goal:</u> $2^{k+1} > k + 1$.

How do you turn the current left side into the desired left side of

2^{k+1}**?** Multiply by 2, so multiply both sides by 2.

$2^{k+1} > 2k$

$= k + k \geq k + 1,$ because $k \geq 1$

$\{2k = k + k \geq k + 1\}$

<u>Conclusion:</u>

$2^n > n$ for all natural numbers, n, by Mathematical Induction.

6. Consider the sequence $\{a_n\}$ with $a_1 = 1$ and $a_{n+1} = \sqrt{2+a_n}$; $n \geq 1$.

a) Show that $a_n < 2$ for all natural numbers, *n*.

<u>Base Step:</u> Show that it's true for $n = 1$.

Left side	**Right side**
$a_1 = \boxed{1}$	**2**

<u>Induction Step:</u> Show that <u>if</u> it's true for $n = k$, then it must also be true for $n = k+1$.

<u>Assuming</u> that it's true for $n = k$, leads to $a_k < 2$.

<u>Goal:</u> $a_{k+1} < 2$.

$a_{k+1} = \sqrt{2+a_k} < \sqrt{2+2} = \sqrt{4} = 2$, because $a_k < 2$ by assumption

<u>Conclusion:</u>

$a_n < 2$ for all natural numbers, *n*, by Mathematical Induction.

b) Show that $a_{n+1} > a_n$ for all natural numbers, *n*.

<u>Base Step:</u> Show that it's true for $n = 1$.

Left side	**Right side**
$a_2 = \boxed{\sqrt{3}}$	$a_1 = \boxed{1}$

<u>Induction Step:</u> Show that <u>if</u> it's true for $n = k$, then it must also be true for $n = k + 1$.

<u>Assuming</u> that it's true for $n = k$, leads to $a_{k+1} > a_k$.

<u>Goal:</u> $a_{k+2} > a_{k+1}$.

$a_{k+2} = \sqrt{2 + a_{k+1}} > \sqrt{2 + a_k} = a_{k+1}$, because $a_{k+1} > a_k$ by assumption

<u>Conclusion:</u>

$a_{n+1} > a_n$ for all natural numbers, *n*, by Mathematical Induction.