Proving More Things Using the Principle of Mathematical Induction:

Divisibility by 3: A whole number, *n*, is divisible by 3, if there is a whole *k* so that $n = 3k$**.**

Must the sum of two whole numbers that are divisible by 3 also be divisible by 3?

Yes, if *n* and *m* are divisible by 3, then $n = 3k$ **and** $m = 3l$**. So their sum** $n + m = 3k + 3l = 3(k + l)$**, which means that** $n + m$ **is divisible by 3.**

Must the product of a whole number and a whole number divisible by 3 be divisible by 3?

Yes, if *n* is divisible by 3, then $n = 3k$**. So the product of *n* with another whole number, *m*,** $mn = 3km$**, which means that** mn **is divisible by 3.**

1. Prove that $4^n - 1$ is divisible by 3 for all natural numbers, *n*.

<u>Base Step:</u> Show that it's true for $n = 1$. $4^1 - 1 = 3$, which is definitely divisible by 3.

<u>Induction Step:</u> Suppose that it's true for $n = k$. So $4^k - 1$ is divisible by 3.

<u>Goal:</u> Show that it's true for $n = k + 1$, i.e. $4^{k+1} - 1$ is divisible by 3.

$$4^{k+1} - 1 = \underbrace{4\left(4^k - 1\right)}_{\text{divisible by 3, by assumption}} + \underbrace{3}_{\text{divisible by 3}}$$

The right side is a sum of numbers divisible by 3, so it's divisible by 3.

<u>Conclusion:</u>

$4^n - 1$ **is divisible by 3 for all natural numbers, *n*, by Mathematical Induction.**

2. Prove that n^3+2n is divisible by 3 for all natural numbers, *n*.

<u>Base Step:</u> Show that it's true for $n=1$.

$1^3+2\cdot 1=1+2=3$, which is definitely divisible by 3.

<u>Induction Step:</u> Suppose that it's true for $n=k$. So k^3+2k is divisible by 3.

<u>Goal:</u> Show that it's true for $n=k+1$, i.e. $(k+1)^3+2(k+1)$ is divisible by 3.

$$(k+1)^3+2(k+1)=k^3+3k^2+3k+1+2k+2$$
$$=(k^3+2k)+(3k^2+3k+3)$$
$$=\underbrace{(k^3+2k)}_{\text{divisible by 3, by assumption}}+\underbrace{3(k^2+k+1)}_{\text{divisible by 3}}$$

The right side is a sum of numbers divisible by 3, so it's divisible by 3.

<u>Conclusion:</u>

n^3+2n **is divisible by 3 for all natural numbers, *n*, by Mathematical Induction.**

Sometimes statements involving natural numbers aren't true for all natural numbers.

Sometimes they're only true for natural numbers *n* with $n \geq n_0$**.**

1. Prove that $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ **for all natural numbers, *n*, with** $n \geq 2$**.**

<u>Base Step:</u> Show that it's true for $n=2$**.**

Left side	**Right side**
$1-\frac{1}{4}=\boxed{\frac{3}{4}}$	$\frac{2+1}{2\cdot 2}=\boxed{\frac{3}{4}}$

Induction Step: **Suppose that it's true for** $n=k$. **So**

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}.$$

Multiply both sides by $\left[1-\frac{1}{(k+1)^2}\right]$ **to get**

$$\underbrace{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{k^2}\right)\left[1-\frac{1}{(k+1)^2}\right]}_{n=k+1}=\frac{k+1}{2k}\left[1-\frac{1}{(k+1)^2}\right]$$

$$=\frac{k+1}{2k}\left[\frac{(k+1)^2}{(k+1)^2}-\frac{1}{(k+1)^2}\right]=\frac{k+1}{2k}\left[\frac{k^2+2k}{(k+1)^2}\right]=\frac{k+1}{2k}\left[\frac{k(k+2)}{(k+1)^2}\right]$$

$=\frac{(k+2)}{2(k+1)}$, which is what you get when you replace n with $k+1$.

Conclusion:

$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ **for all natural numbers, *n*, with** $n\geq 2$**, by Mathematical Induction.**

2. Prove that $n^2>n+1$ **for all natural numbers, *n*, with** $n\geq 2$**.**

Base Step: **Show that it's true for** $n=2$**.**

Left side	**Right side**
$2^2=\boxed{4}$	$2+1=\boxed{3}$

Induction Step: **Suppose that it's true for $n=k$. So $k^2 > k+1$.**

$$(k+1)^2 = k^2 + 2k + 1 > (k+1) + 1 + 2k > (k+1) + 1 = k + 2$$

Because $2k > 0$.

Conclusion:

$n^2 > n+1$ **for all natural numbers, *n*, with $n \geq 2$, by Mathematical Induction.**

Factorials:

$$\begin{aligned} n! &= 1\cdot 2\cdot 3\cdot\dots\cdot n \\ &= n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 2\cdot 1 \end{aligned}$$

$$1! = 1$$

$$2! = 2\cdot 1 = 2$$

$$3! = 3\cdot 2\cdot 1 = 6$$

$$4! = 4\cdot 3\cdot 2\cdot 1 = 4\cdot 3! = 4\cdot 6 = 24$$

$$5! = 5\cdot 4! = 120$$

$$6! = 6\cdot 5! = 720$$

By special separate definition, $0! = 1$**.**

3. Prove that $n! > n^2$ for all natural numbers, *n*, with $n \geq 4$.

<u>Base Step:</u> Show that it's true for $n = 4$.

Left side	**Right side**
$4! = \boxed{24}$	$4^2 = \boxed{16}$

<u>Induction Step:</u> Suppose that it's true for $n = k$. So $k! > k^2$. Multiply both sides of the inequality by $(k+1)$, to get

$$(k+1)! > \underbrace{k^2(k+1) > (k+1)(k+1)}_{\text{from the previous induction problem}} = (k+1)^2$$

<u>Conclusion:</u>

$n! > n^2$ for all natural numbers, *n*, with $n \geq 4$, by Mathematical Induction.

Sometimes statements that can be proven by induction can also be proven in another way.

1. Prove that $n^3 + 3n^2 + 2n$ **is divisible by 3 for all natural numbers, *n*.**

$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n+1)(n+2)$. **Every third natural number starting with 1 is a multiple of 3, so if you have three consecutive natural numbers, one of them must be a multiple of 3, and hence their product is a multiple of 3.**

2. Prove that $n^3 + 3n^2 + 2n$ **is divisible by 6 for all natural numbers, *n*.**

$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n+1)(n+2)$. **Every second natural number starting with 1 is a multiple of 2, so if you have three consecutive natural numbers, at least one of them must be a multiple of 2, and hence their product is a multiple of 2 and 3. This means their product must be a multiple of 6.**

3. Prove that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdot\cdots\cdot\left(1-\frac{1}{n}\right)=\frac{1}{n}$ **for all natural numbers, *n*, with** $n \geq 2$**.**

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdot\cdots\cdot\left(1-\frac{1}{n}\right)=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{4}}\cdot\cdots\cdot\frac{\cancel{n-2}}{\cancel{n-1}}\cdot\frac{\cancel{n-1}}{n}=\frac{1}{n}$$