Binomial Expansions:

Multiplying out a power of a binomial is called expanding the binomial (power).

$$(x+a)^0 = 1$$

$$(x+a)^1 = x+a$$

$$(x+a)^2 = x^2 + 2ax + a^2$$

$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$

If we just look at the coefficients of the terms in the expansions so far, here's what we get:

$(x+a)^0$:				**1**			
$(x+a)^1$:			**1**		**1**		
$(x+a)^2$:		**1**		**2**		**1**	
$(x+a)^3$:	**1**		**3**		**3**		**1**

This is the beginning of a triangle of coefficients called Pascal's Triangle.

There is a nice additive pattern to the coefficients.

Let's see why it's true in going from $(x+a)^2 = 1x^2 + 2ax + 1a^2$ **to** $(x+a)^3$**.**

$$(x+a)^3 = (x+a)(x+a)^2 = (x+a)(x^2+2ax+a^2)$$

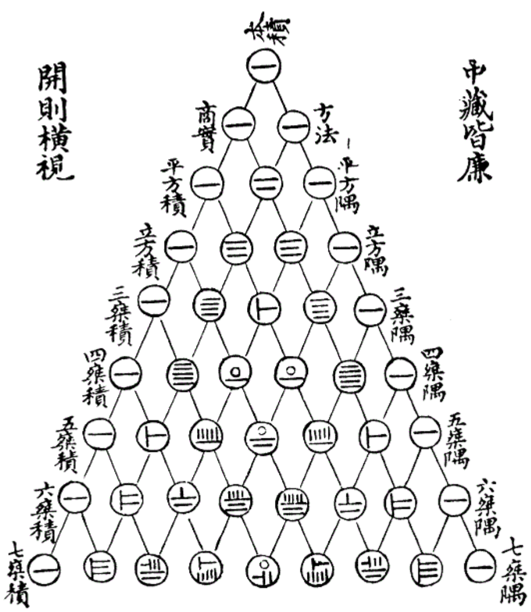
$$= x^3 + 2ax^2 + a^2x + ax^2 + 2a^2x + a^3$$

$$= 1x^3 + (1+2)ax^2 + (2+1)a^2x + 1a^3$$

1 2 1

1 1

1



Let's continue Pascal's Triangle a little further.

$(x+a)^0$:						1					
$(x+a)^1$:					1		1				
$(x+a)^2$:				1		2		1			
$(x+a)^3$:			1		3		3		1		
$(x+a)^4$:		1		4		6		4		1	
$(x+a)^5$:	1		5		10		10		5		1

If you look at the powers of *x* and *a* in the terms, there is another pattern. The powers of *x* decrease by 1 from left to right, while the powers of *a* increase by 1 from left to right. Also the sum of the powers is always equal to the exponent on the binomial.

Let's use Pascal's Triangle to complete the expansion of the binomial $(x+1)^4$.

$$(x+1)^4 = \boxed{1}x^4 \cdot 1^0 + \boxed{4}x^3 \cdot 1^1 + \boxed{6}x^2 \cdot 1^2 + \boxed{4}x^1 \cdot 1^3 + \boxed{1}x^0 \cdot 1^4$$

$$= x^4 + 4x^3 + 6x^2 + 4x + 1$$

Let's use Pascal's Triangle to complete the expansion of the binomial $(x+2)^3$.

$$(x+2)^3 = \boxed{1}x^3 \cdot 2^0 + \boxed{3}x^2 \cdot 2^1 + \boxed{3}x^1 \cdot 2^2 + \boxed{1}x^0 \cdot 2^3$$

$$= x^3 + 6x^2 + 12x + 8$$

Let's use Pascal's Triangle to complete the expansion of the binomial $(x-1)^5$.

$$(x-1)^5 = \boxed{1}x^5\cdot(-1)^0 + \boxed{5}x^4\cdot(-1)^1 + \boxed{10}x^3\cdot(-1)^2 + \boxed{10}x^2\cdot(-1)^3 + \boxed{5}x^1\cdot(-1)^4 + \boxed{1}x^0\cdot(-1)^5$$

$$= x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$$

Let's use Pascal's Triangle to complete the expansion of the binomial $(3x-2)^4$.

$$(3x-2)^4 = \boxed{1}(3x)^4\cdot(-2)^0 + \boxed{4}(3x)^3\cdot(-2)^1 + \boxed{6}(3x)^2\cdot(-2)^2 + \boxed{4}(3x)^1\cdot(-2)^3 + \boxed{1}(3x)^0\cdot(-2)^4$$

$$= 81x^4 - 216x^3 + 216x^2 - 96x - 16$$

Pascal's Triangle is a nice way to generate the coefficients in a binomial expansion, but unfortunately, it finds them recursively. There is a direct way of finding the coefficients using a formula that involves factorials.

$$\begin{aligned} n! &= 1\cdot 2\cdot 3\cdot\cdots\cdot n \\ &= n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot 2\cdot 1 \end{aligned}$$

$$1! = 1$$

$$2! = 2\cdot 1 = 2$$

$$3! = 3\cdot 2\cdot 1 = 6$$

$$4! = 4\cdot 3\cdot 2\cdot 1 = 4\cdot 3! = 4\cdot 6 = 24$$

$$5! = 5\cdot 4! = 120$$

$$6! = 6\cdot 5! = 720$$

By special separate definition, $0! = 1$**.**

Binomial Coefficient Formula:

$$\binom{n}{j} = \frac{n!}{j! \cdot (n-j)!} \quad \text{on calculators: } {}_nC_j$$

1. $\binom{5}{0} = \frac{5!}{0! \cdot 5!} = 1$

$\left\{ \binom{n}{0} \right\} = 1$

2. $\binom{5}{1} = \frac{5!}{1! \cdot 4!} = 5$

$\left\{ \binom{n}{1} = n \right\}$

3. $\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$

4. $\binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10$

5. $\binom{5}{4} = \frac{5!}{4! \cdot 1!} = 5$

$\left\{ \binom{n}{n-1} = n \right\}$

6. $\binom{5}{5} = \frac{5!}{5! \cdot 0!} = 1$

$\left\{ \binom{n}{n} = 1 \right\}$

Binomial Theorem:

$$\begin{aligned}(x+a)^n &= \binom{n}{0}x^n a^0 + \binom{n}{1}x^{n-1}a^1 + \binom{n}{2}x^{n-2}a^2 + \cdots + \binom{n}{n-1}x^1 a^{n-1} + \binom{n}{n}x^0 a^n \\ &= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}a^j\end{aligned}$$

When using the Binomial Theorem, you may use the binomial coefficient formula or Pascal's Triangle to find the values of the coefficients.

$$(x+1)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5 + \binom{6}{2}x^4 + \binom{6}{3}x^3 + \binom{6}{4}x^2 + \binom{6}{5}x + \binom{6}{6}$$

1 6 15 20 15 6 1

						1						
					1		**1**					
				1		**2**		**1**				
			1		**3**		**3**		**1**			
		1		**4**		**6**		**4**		**1**		
	1		**5**		**10**		**10**		**5**		**1**	
1		**6**		**15**		**20**		**15**		**6**		**1**

$$(x+1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

Examples:

1. Find the coefficient of x^3 **in the expansion of** $(x+3)^{10}$**.**

From the Binomial Theorem, the term containing x^3 **is given by** $\binom{10}{7}x^{10-7}3^7$**, so the coefficient is** $\binom{10}{7}3^7 = 262{,}440$**.**

2. Find the coefficient of x^2 **in the expansion of** $(2x-3)^9$**.**

From the Binomial Theorem, the term containing x^2 **is given by** $\binom{9}{7}(2x)^{9-7}(-3)^7$**, so the coefficient is** $\binom{9}{7}2^2(-3)^7 = -314{,}928$**.**

3. Find the sixth term in the expansion of $(3x+2)^8$.

From the Binomial Theorem, the sixth term is given by $\binom{8}{5}(3x)^{8-5}2^5 = 48{,}384x^3$.

{The reason that 5 is used is because the first term corresponds to j = 0.}

4. Find the coefficient of x^0 in the expansion of $\left(x-\dfrac{1}{x^2}\right)^9$.

From the Binomial Theorem, a term is given by

$\binom{9}{j}x^{9-j}\left(-\dfrac{1}{x^2}\right)^j = \binom{9}{j}x^{9-j}(-1)^j x^{-2j} = \binom{9}{j}(-1)^j x^{9-3j}$. **The value of *j* that gives *x* the power zero is 3, so the coefficient is** $\binom{9}{3}(-1)^3 = -84$

5. Find the coefficient of x^2 in the expansion of $\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)^8$.

From the Binomial Theorem, a term is given by

$\binom{8}{j}\left(\sqrt{x}\right)^{8-j}\left(\frac{3}{\sqrt{x}}\right)^j=\binom{8}{j}x^{4-\frac{1}{2}j}\cdot 3^j\cdot x^{-\frac{1}{2}j}=\binom{8}{j}3^j x^{4-j}$. **The value of *j* that gives *x* the power 2 is 2, so the coefficient is** $\binom{8}{2}3^2=252$.

6. Find the exact value of $\binom{1{,}000}{0}+\binom{1{,}000}{1}+\binom{1{,}000}{2}+\cdots+\binom{1{,}000}{1{,}000}$.

What values of *x*, *a*, and *n* in $(x+a)^n$ would produce this expansion?

$$\left\{\sum_{j=0}^{1{,}000}\binom{1{,}000}{j}\boxed{1}^{1{,}000-j}\boxed{1}^{j}=\left(\boxed{1}+\boxed{1}\right)^{1{,}000}\right\}$$

$$(1+1)^{1{,}000}=2^{1{,}000}$$

7. Find the exact value of $\binom{1{,}000}{0}-\binom{1{,}000}{1}+\binom{1{,}000}{2}-\binom{1{,}000}{3}+\cdots+\binom{1{,}000}{1{,}000}$.

What values of *x*, *a*, and *n* in $(x+a)^n$ would produce this expansion?

$$\left\{\sum_{j=0}^{1{,}000}\binom{1{,}000}{j}\boxed{1}^{1{,}000-j}\boxed{-1}^{j}=\left(\boxed{1}+\boxed{-1}\right)^{1{,}000}\right\}$$

$$(1-1)^{1{,}000}=0^{1{,}000}=0$$