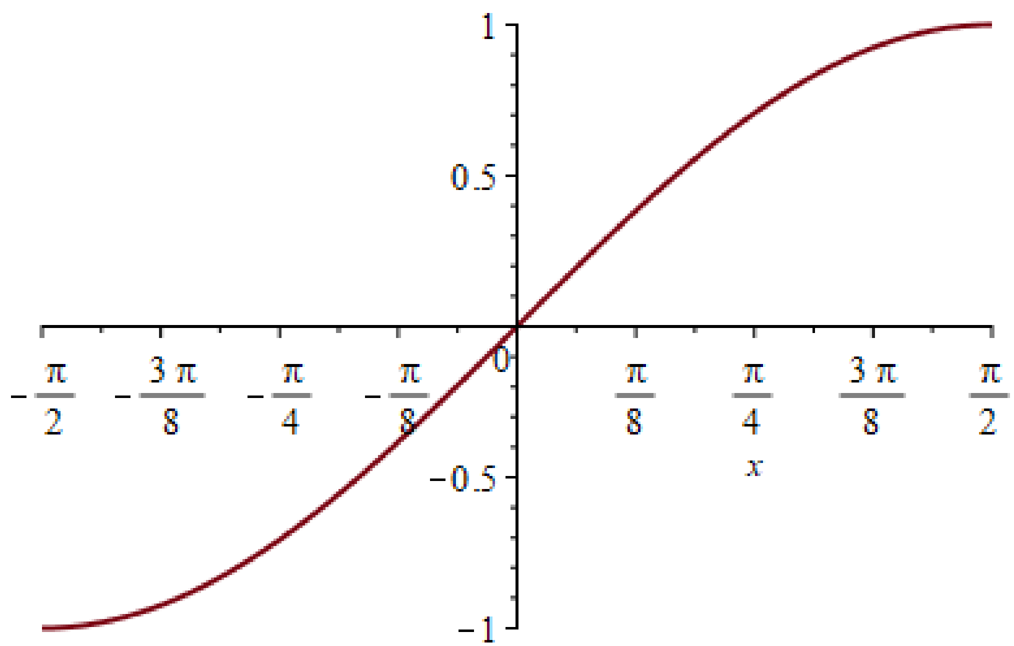
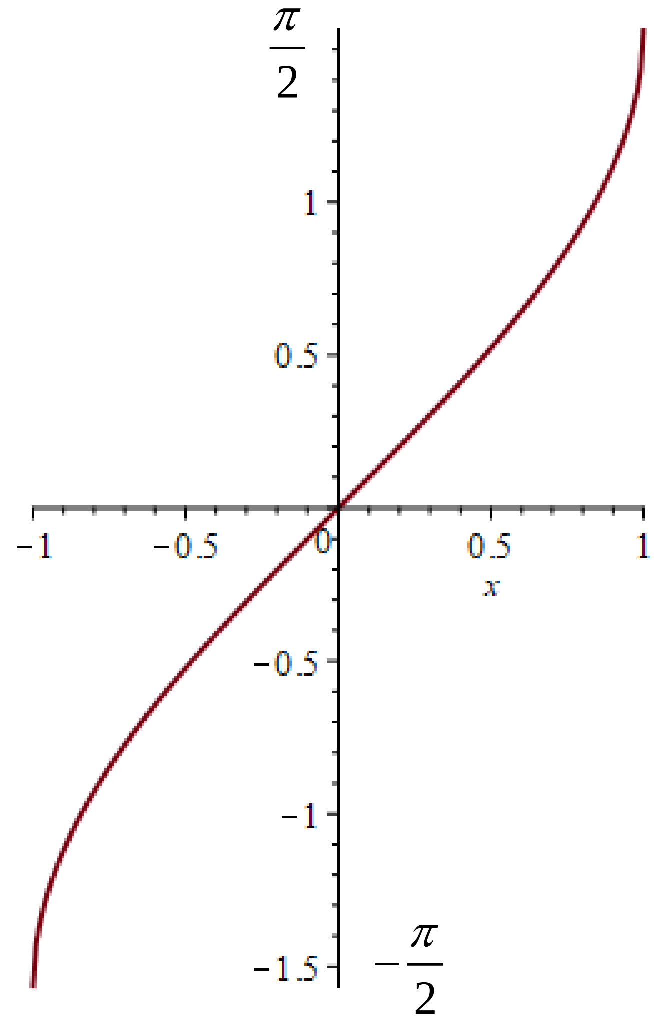
Review of Inverse Trigonometric Functions:

None of the Trigonometric Functions is 1-1, so none of them have inverses without restricting their domains.

<u>Inverse Sine:</u> The restriction on the domain of the sine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



The graph of the inverse sine function is found by reflection about the line $y = x$.



$sin^{-1} x$ **is the angle in** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **whose sine is *x*.**

Examples:

1. $\sin^{-1}(-1)$

$$-\frac{\pi}{2}$$

2. $\sin^{-1}\left(-\frac{1}{2}\right)$

$$-\frac{\pi}{6}$$

3. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$\frac{\pi}{3}$$

4. $\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

$$\frac{1}{3} \quad , \sin\left(\sin^{-1}(x)\right) = x \text{ for } -1 \le x \le 1$$

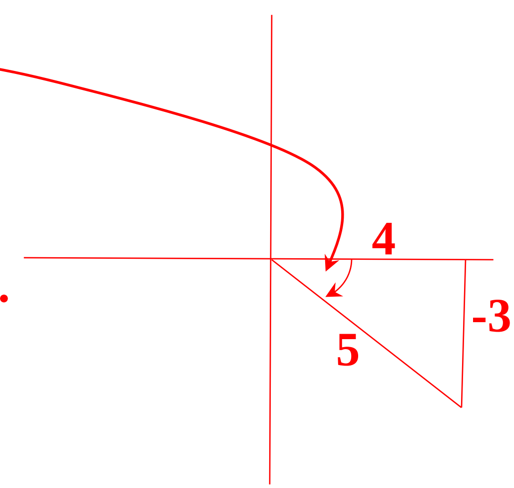
5. $\sin^{-1}\left(\sin\left(\frac{6\pi}{7}\right)\right)$

$$\frac{\pi}{7}$$

It's not $\frac{6\pi}{7}$ because it's not in the range of the inverse sine.

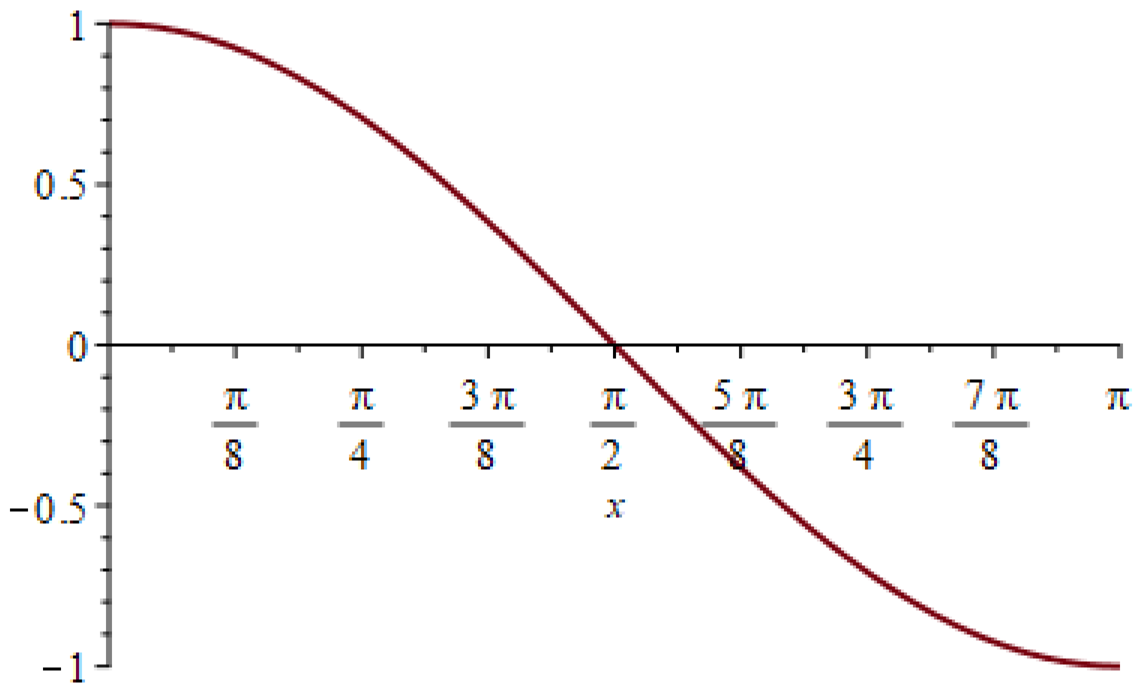
6. $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

Cosine is $\frac{adj}{hyp}$**, so** $\frac{4}{5}$**.**

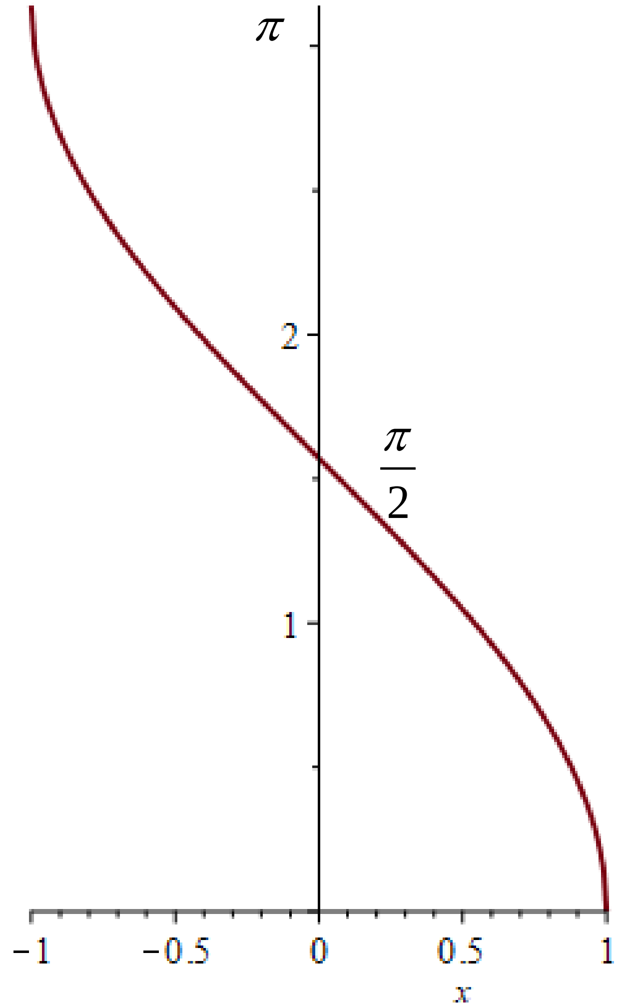


<u>Inverse Cosine:</u> **The restriction on the domain of the cosine function is $[0,\pi]$.**

cosine



The graph of the inverse cosine function is found by reflection about the line $y = x$.



$\cos^{-1} x$ **is the angle in $[0, \pi]$ whose cosine is *x*.**

Examples:

1. $cos^{-1}(1)$

0

2. $cos^{-1}\left(-\frac{1}{2}\right)$

$\frac{2\pi}{3}$

3. $cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

$\frac{3\pi}{4}$

4. $cos\left(cos^{-1}\left(-\frac{1}{3}\right)\right)$

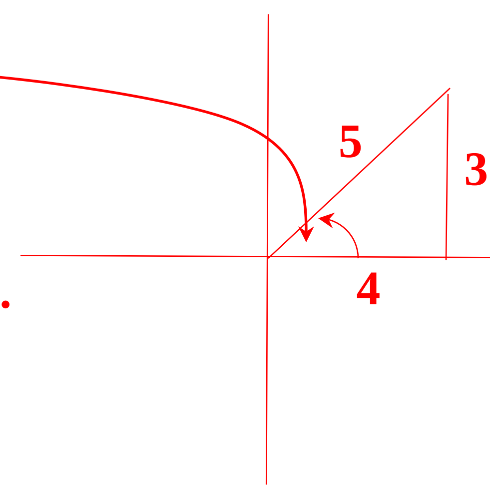
$-\frac{1}{3}$, $\cos\left(\cos^{-1}(x)\right) = x$ for $-1 \le x \le 1$

5. $cos^{-1}\left(cos\left(-\frac{\pi}{4}\right)\right)$

$\frac{\pi}{4}$

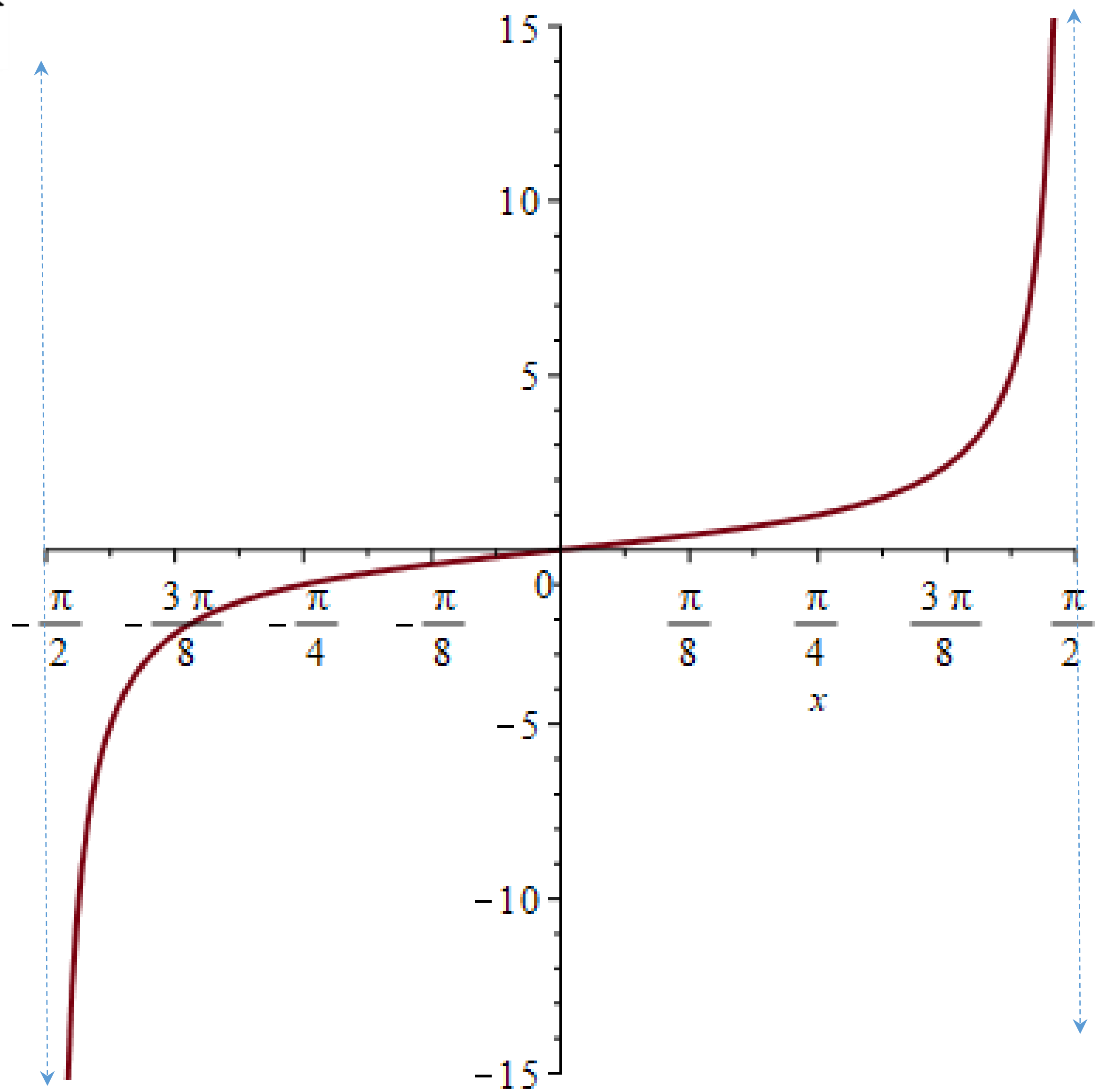
It’s not $-\frac{\pi}{4}$ because it’s not in the range of the inverse cosine.

6. $sin\left(cos^{-1}\left(\frac{4}{5}\right)\right)$

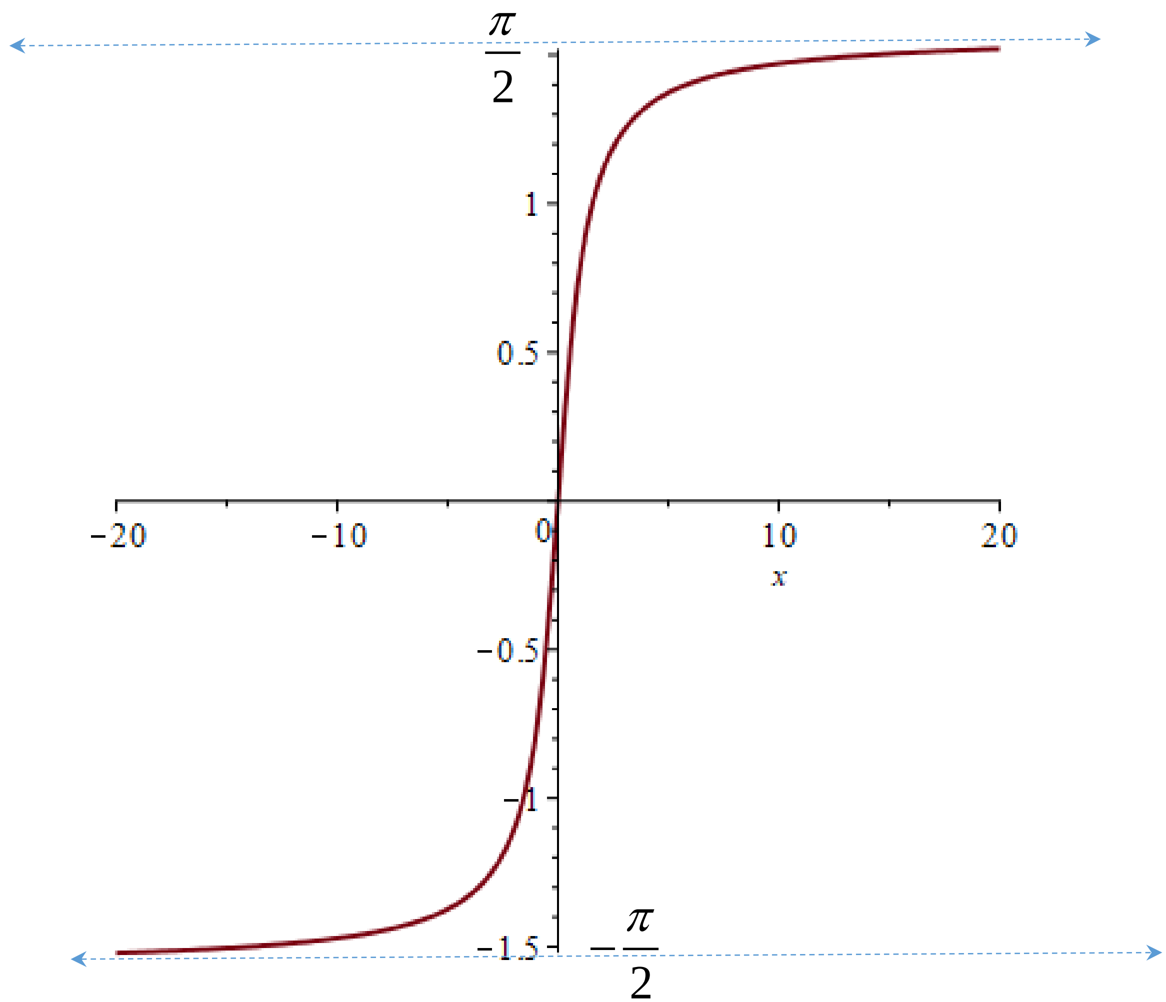


Sine is $\frac{opp}{hyp}$, so $\frac{3}{5}$.

<u>Inverse Tangent:</u> The restriction on the domain of the tangent function is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.



The graph of the inverse tangent function is found by reflection about the line $y = x$.



$tan^{-1} x$ **is the angle in** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **whose tangent is** ***x*****.**

Examples:

1. $tan^{-1}(-1)$

$-\frac{\pi}{4}$

2. $tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$-\frac{1}{\sqrt{3}} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \Rightarrow$ sine is $-\frac{1}{2}$**, so** $-\frac{\pi}{6}$

3. $tan^{-1}\left(\sqrt{3}\right)$

$\sqrt{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \Rightarrow$ sine is $\frac{\sqrt{3}}{2}$**, so** $\frac{\pi}{3}$

4. $tan\left(tan^{-1}\left(-\frac{1}{3}\right)\right)$

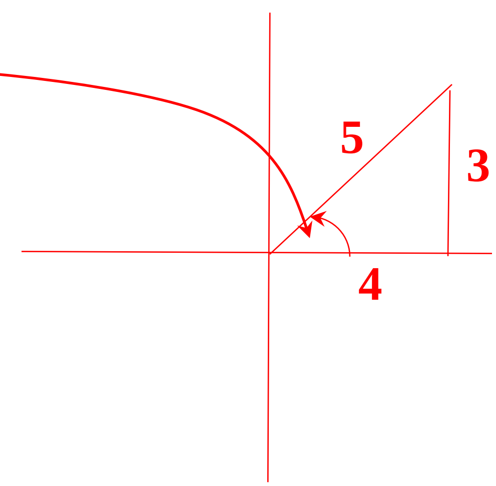
$-\frac{1}{3}$, $\tan\left(\tan^{-1}(x)\right) = x$

5. $tan^{-1}\left(tan\left(\frac{4\pi}{5}\right)\right)$

$-\frac{\pi}{5}$

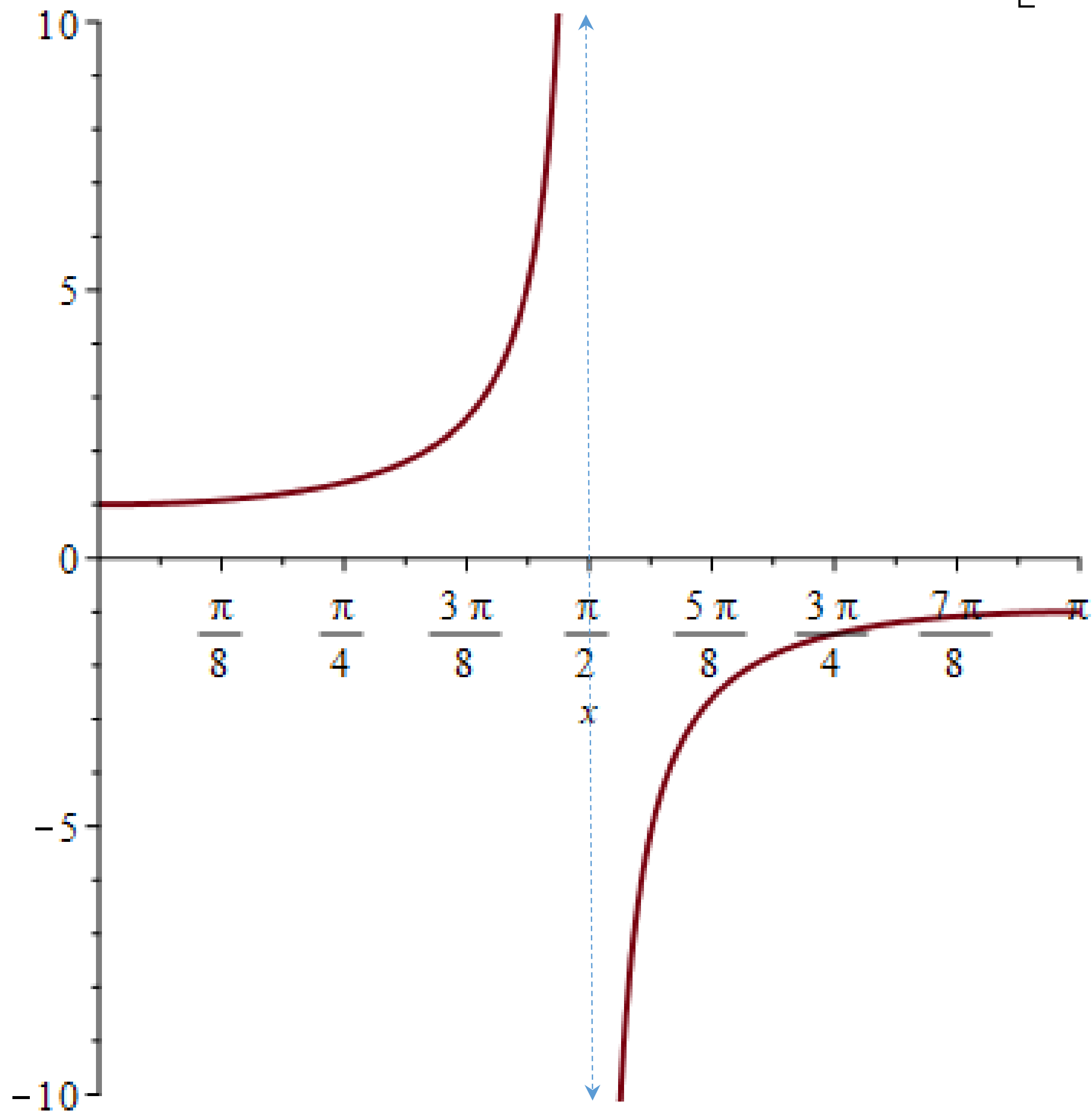
It's not $\frac{4\pi}{5}$ because it's not in the range of the inverse tangent.

6. $sin\left(tan^{-1}\left(\frac{3}{4}\right)\right)$

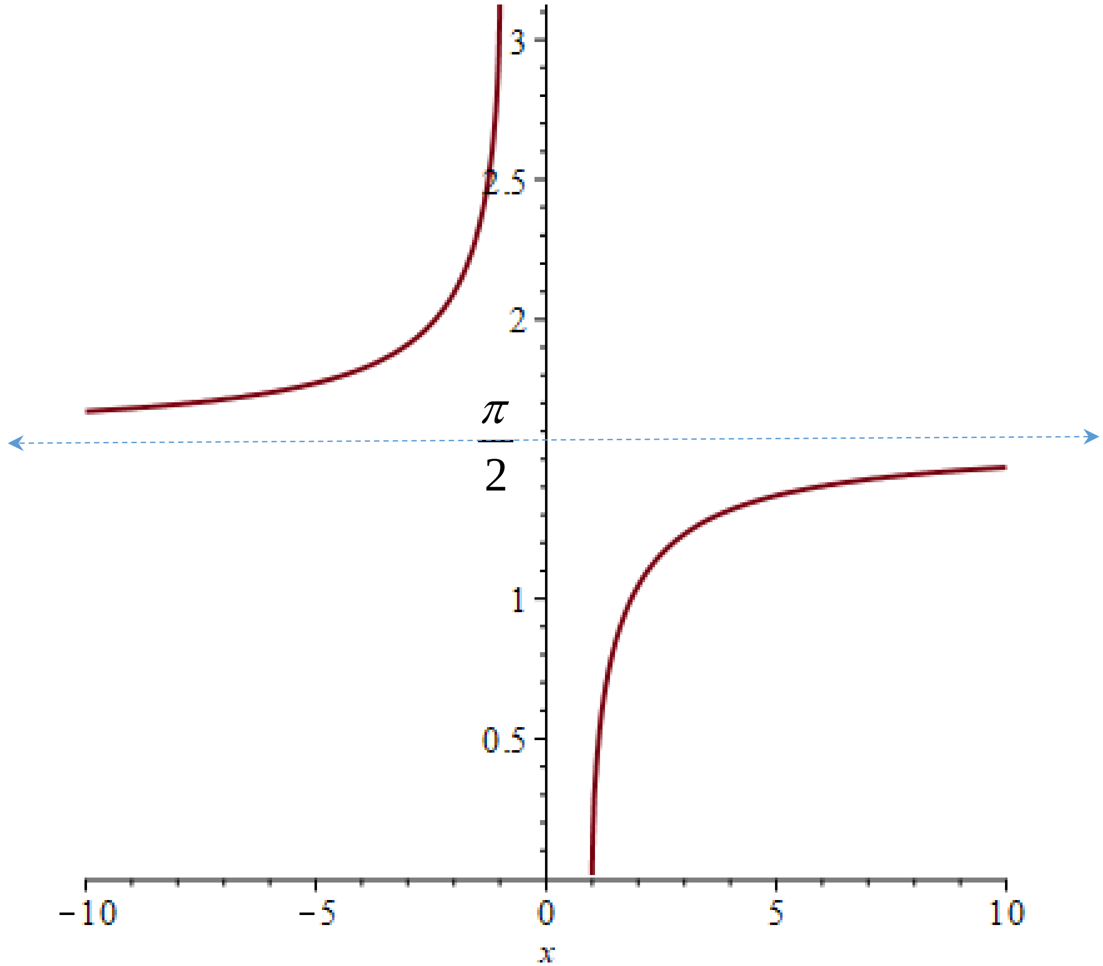


Sine is $\frac{opp}{hyp}$**, so** $\frac{3}{5}$**.**

<u>Inverse Secant:</u> The restriction on the domain of the secant function is $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ **secant** .



The graph of the inverse secant function is found by reflection about the line $y = x$**.**



$sec^{-1}x$ **is the angle in** $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ **whose secant is *x*.**

Examples:

1. $sec^{-1}(-1)$

Cosine must be -1, so π.

2. $sec^{-1}\left(\sqrt{2}\right)$

Cosine must be $\frac{1}{\sqrt{2}}$, so $\frac{\pi}{4}$.

3. $sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Cosine must be $\frac{\sqrt{3}}{2}$, so $\frac{\pi}{6}$.

4. $sec\left(sec^{-1}(-3)\right)$

$-3 \quad , \sec\left(\sec^{-1}(x)\right) = x$ for $x \geq 1$ or $x \leq -1$

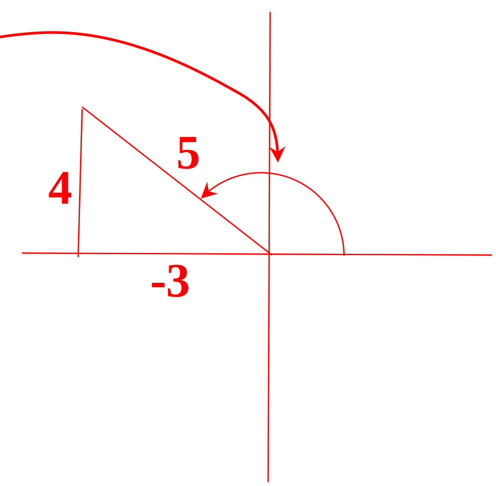
5. $sec^{-1}\left(sec\left(-\frac{2\pi}{9}\right)\right)$

$$\frac{2\pi}{9}$$

It's not $-\frac{2\pi}{9}$ because it's not in the range of the inverse secant.

6. $sin\left(sec^{-1}\left(-\frac{5}{3}\right)\right)$

Sine is $\frac{opp}{hyp}$, so $\frac{4}{5}$.



Inverse Cotangent: **The restriction on the domain of the cotangent function is** $(0,\pi)$**.**

$$cot^{-1}(1) = \frac{\pi}{4}$$

$$cot^{-1}(-1) = \frac{3\pi}{4}$$

Inverse Cosecant: **The restriction on the domain of the cosecant function is** $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$**.**

$$csc^{-1}(2)$$

Sine must be $\frac{1}{2}$**, so** $\frac{\pi}{6}$**.**

$$csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$$

Sine must be $-\frac{\sqrt{3}}{2}$**, so** $-\frac{\pi}{3}$**.**

Trigonometric Equations:

Because of the periodicity of the trigonometric functions, trig. equations will generally have infinitely many solutions. Unless it's stated otherwise, we'll just find the solutions in the interval $\left[0,2\pi\right)$.

Examples:

1. $2\sin\theta + 3 = 2$

$$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{7\pi}{6} + 2\pi, \frac{7\pi}{6} + 4\pi, \ldots \text{ or } \theta = \frac{11\pi}{6}, \frac{11\pi}{6} + 2\pi, \frac{11\pi}{6} + 4\pi, \ldots$$

The only values in the interval $\left[0,2\pi\right)$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$**.**

2. $\tan^2\theta = 1$

$$\tan\theta = \pm 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{\pi}{4} + 2\pi, \ldots \text{ or } \theta = \frac{5\pi}{4}, \frac{5\pi}{4} + 2\pi, \ldots \text{ or } \theta = \frac{3\pi}{4}, \frac{3\pi}{4} + 2\pi, \ldots \text{ or } \theta = \frac{7\pi}{4}, \frac{7\pi}{4} + 2\pi, \ldots$$

The only values in the interval $\left[0,2\pi\right)$ are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$**.**

3. $\cos 2\theta = -1$

$$2\theta = \pi, \pi + 2\pi, \pi + 4\pi, \ldots \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

The only values in the interval $[0, 2\pi)$ are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

4. $csc\,\frac{\theta}{3} = \frac{2}{\sqrt{3}}$

$$\frac{\theta}{3} = \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, \ldots \text{ or } \frac{\theta}{3} = \frac{2\pi}{3}, \frac{2\pi}{3} + 2\pi, \ldots \Rightarrow \theta = \pi, 7\pi, \ldots \text{ or } \theta = 2\pi, 8\pi, \ldots$$

The only value in the interval $[0, 2\pi)$ is π.

5. $sin\left(2\theta-\frac{\pi}{2}\right)=-1$

$$2\theta-\frac{\pi}{2}=\frac{3\pi}{2},\frac{3\pi}{2}+2\pi,\frac{3\pi}{2}+4\pi,\ldots \text{ or } 2\theta-\frac{\pi}{2}=\frac{3\pi}{2}-2\pi,\frac{3\pi}{2}-4\pi,\ldots \Rightarrow \theta=\pi,2\pi,\ldots \text{ or } \theta=0,-\pi,\ldots$$

The only values in the interval $[0,2\pi)$ **are** 0 and π**.**

6. $2sin^2\theta-sin\theta=0$

Factor into $sin\theta(2sin\theta-1)=0\Rightarrow 2sin\theta\left(sin\theta-\frac{1}{2}\right)=0\Rightarrow sin\theta=0$ or $sin\theta=\frac{1}{2}$

$$\Rightarrow \theta=0,\pi,2\pi,\ldots \text{ or } \theta=\frac{\pi}{6},\frac{\pi}{6}+2\pi,\ldots \text{ or } \theta=\frac{5\pi}{6},\frac{5\pi}{6}+2\pi,\ldots$$

The only values in the interval $[0,2\pi)$ **are** $0,\pi,\frac{\pi}{6},\frac{5\pi}{6}$**.**

7. $2cos^2\theta + cos\theta - 1 = 0$

Factor into $(2cos\theta - 1)(cos\theta + 1) = 0 \Rightarrow cos\theta = -1$ or $\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \pi, 3\pi, \ldots$ or $\theta = \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, \ldots$ or $\theta = \frac{5\pi}{3}, \frac{5\pi}{3} + 2\pi, \ldots$

The only values in the interval $[0, 2\pi)$ **are** $\pi, \frac{\pi}{3}, \frac{5\pi}{3}$**.**

8. $sin^2\theta - cos^2\theta = 1 + cos\theta$ ***{Trig. identity:*** $sin^2\theta = 1 - cos^2\theta$***.}***

$1 - 2cos^2\theta = 1 + cos\theta \Rightarrow 2cos^2\theta + cos\theta = 0 \Rightarrow 2cos\theta\left(cos\theta + \frac{1}{2}\right) = 0 \Rightarrow cos\theta = 0$ or $cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \ldots$ or $\theta = \frac{3\pi}{2}, \frac{3\pi}{2} + 2\pi, \ldots$ or $\theta = \frac{2\pi}{3}, \frac{2\pi}{3} + 2\pi, \ldots$ or $\theta = \frac{4\pi}{3}, \frac{4\pi}{3} + 2\pi, \ldots$

The only values in the interval $[0, 2\pi)$ **are** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$**.**

9. $\cos\theta - \sin(-\theta) = 0$ **{*Trig. identity:*** $\sin(-\theta) = -\sin\theta$**.}**

$\cos\theta + \sin\theta = 0 \Rightarrow \cos\theta = -\sin\theta \Rightarrow \theta = \frac{3\pi}{4}, \frac{3\pi}{4} + 2\pi, \ldots$ or $\theta = \frac{7\pi}{4}, \frac{7\pi}{4} + 2\pi, \ldots$

The only values in the interval $[0, 2\pi)$ **are** $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$**.**

10. $2\sin^2\theta - 5\sin\theta + 2 = 0$

Factor into $(2\sin\theta - 1)(\sin\theta - 2) = 0 \Rightarrow \sin\theta = \frac{1}{2}$ or $\sin\theta = 2 > 1$(Not Possible!)

$\Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{6} + 2\pi, \ldots$ or $\theta = \frac{5\pi}{6}, \frac{5\pi}{6} + 2\pi, \ldots$

The only values in the interval $[0, 2\pi)$ **are** $\frac{\pi}{6}$ and $\frac{5\pi}{6}$**.**

11. $3sin^2\theta + 2sin\theta - 1 = 0$

Factor into $(3sin\theta - 1)(sin\theta + 1) = 0 \Rightarrow sin\theta = \frac{1}{3}$ or $\sin\theta = -1$

$\Rightarrow \theta = \frac{3\pi}{2}, \frac{3\pi}{2} + 2\pi, \ldots$ or $\theta = sin^{-1}\left(\frac{1}{3}\right), sin^{-1}\left(\frac{1}{3}\right) + 2\pi, \ldots$ or $\theta = \pi - sin^{-1}\left(\frac{1}{3}\right), \left[\pi - sin^{-1}\left(\frac{1}{3}\right)\right] + 2\pi, \ldots$

The only values in the interval $[0, 2\pi)$ are $\frac{3\pi}{2}, sin^{-1}\left(\frac{1}{3}\right), \pi - sin^{-1}\left(\frac{1}{3}\right)$.

12. $sin\theta - \sqrt{3}cos\theta = 1$ **{Square both sides, and use $sin^2\theta + cos^2\theta = 1$.}**

Square to get $sin^2\theta - 2\sqrt{3}\,sin\theta\,cos\theta + 3cos^2\theta = 1 \Rightarrow 2cos^2\theta - 2\sqrt{3}\,sin\theta\,cos\theta = 0$

$\Rightarrow 2cos\theta\left(cos\theta - \sqrt{3}\,sin\theta\right) = 0 \Rightarrow cos\theta = 0$ or $cos\theta - \sqrt{3}\,sin\theta = 0 \Rightarrow cos\theta = 0$ or $tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \ldots$ or $\theta = \frac{3\pi}{2}, \frac{3\pi}{2} + 2\pi, \ldots$ or $\theta = \frac{\pi}{6}, \frac{\pi}{6} + 2\pi, \ldots$ or $\theta = \frac{7\pi}{6}, \frac{7\pi}{6} + 2\pi, \ldots$

The only *possible* solutions in the interval $[0, 2\pi)$ are $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{7\pi}{6}$.

The only ones that actually are solutions are $\dfrac{\pi}{2}$ and $\dfrac{7\pi}{6}$**.**

13. $sin 2\theta = cos\theta$ **{Trig. identity:** $sin 2\theta = 2 sin\theta cos\theta$**.}**

$$2 sin\theta cos\theta = cos\theta \Rightarrow 2 sin\theta cos\theta - cos\theta = 0 \Rightarrow 2 cos\theta\left(sin\theta - \frac{1}{2}\right) = 0 \Rightarrow cos\theta = 0 \text{ or } sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \ldots \text{ or } \theta = \frac{3\pi}{2}, \frac{3\pi}{2} + 2\pi, \ldots \text{ or } \theta = \frac{\pi}{6}, \frac{\pi}{6} + 2\pi, \ldots \text{ or } \theta = \frac{5\pi}{6}, \frac{5\pi}{6} + 2\pi, \ldots$$

The only values in the interval $\left[0, 2\pi\right)$ **are** $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$**.**

14. $cos 2\theta + 6 sin^2\theta = 4$

{Trig. identity: $cos 2\theta = cos^2\theta - sin^2\theta$ or $2cos^2\theta - 1$ or $1 - 2sin^2\theta$**.}**

$$1 + 4 sin^2\theta = 4 \Rightarrow sin^2\theta = \frac{3}{4} \Rightarrow sin\theta = \pm\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, \ldots \text{ or } \theta = \frac{2\pi}{3}, \frac{2\pi}{3} + 2\pi, \ldots$$

$$\text{or } \theta = \frac{4\pi}{3}, \frac{4\pi}{3} + 2\pi, \ldots \text{ or } \theta = \frac{5\pi}{3}, \frac{5\pi}{3} + 2\pi, \ldots$$

The only values in the interval $[0,2\pi)$ are $\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$.

15. $sin2\theta - sin4\theta = 0$ ***{Trig. identity:*** $sin4\theta = 2sin2\theta cos2\theta$***.}***

$sin2\theta - 2sin2\theta cos2\theta = 0 \Rightarrow sin2\theta(1-2cos2\theta) = 0 \Rightarrow sin2\theta = 0$ or $cos2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = 0,\pi,2\pi,3\pi,\ldots$ or $2\theta = \frac{\pi}{3},\frac{\pi}{3}+2\pi,\frac{\pi}{3}+4\pi,\ldots$ or $2\theta = \frac{5\pi}{3},\frac{5\pi}{3}+2\pi,\frac{5\pi}{3}+4\pi,\ldots$

$\Rightarrow \theta = 0,\frac{\pi}{2},\pi,\frac{3\pi}{2},\ldots$ or $\theta = \frac{\pi}{6},\frac{\pi}{6}+\pi,\frac{\pi}{6}+2\pi,\ldots$ or $\theta = \frac{5\pi}{6},\frac{5\pi}{6}+\pi,\frac{5\pi}{6}+2\pi,\ldots$

The only values in the interval $[0,2\pi)$ are $0,\frac{\pi}{2},\pi,\frac{3\pi}{2},\frac{\pi}{6},\frac{7\pi}{6},\frac{5\pi}{6},\frac{11\pi}{6}$.

16. $cos^2\theta + 2cos\theta - 4 = 0$

From the quadratic formula $cos\theta = \frac{-2\pm\sqrt{4+16}}{2} \Rightarrow cos\theta = \frac{-2\pm2\sqrt{5}}{2} \Rightarrow cos\theta = -1\pm\sqrt{5}$

$\Rightarrow cos\theta = -1+\sqrt{5} > 1$(Not possible!) or $cos\theta = -1-\sqrt{5} < -1$(Not possible!)

So there's no solution.

17. $sin^{14}\theta + cos^{8}\theta + 1 = 4$

$$0 \le sin^{14}\theta \le 1, \; 0 \le cos^{8}\theta \le 1$$

So $\boxed{0} \le sin^{14}\theta + cos^{8}\theta \le \boxed{2}$

So $\boxed{1} \le sin^{14}\theta + cos^{8}\theta + 1 \le \boxed{3}$

So?

The left-side can't possibly equal 4, so there's no solution.

18. $\left(1+\sin^{12}\theta\right)\left(2-\cos^{3}\theta\right)=7$

$$1\le 1+\sin^{12}\theta\le 2,\ 1\le 2-\cos^{3}\theta\le 3$$

So $\boxed{1}\le\left(1+\sin^{12}\theta\right)\left(2-\cos^{3}\theta\right)\le\boxed{6}$

So?

The left-side can’t possibly equal 7, so there’s no solution.