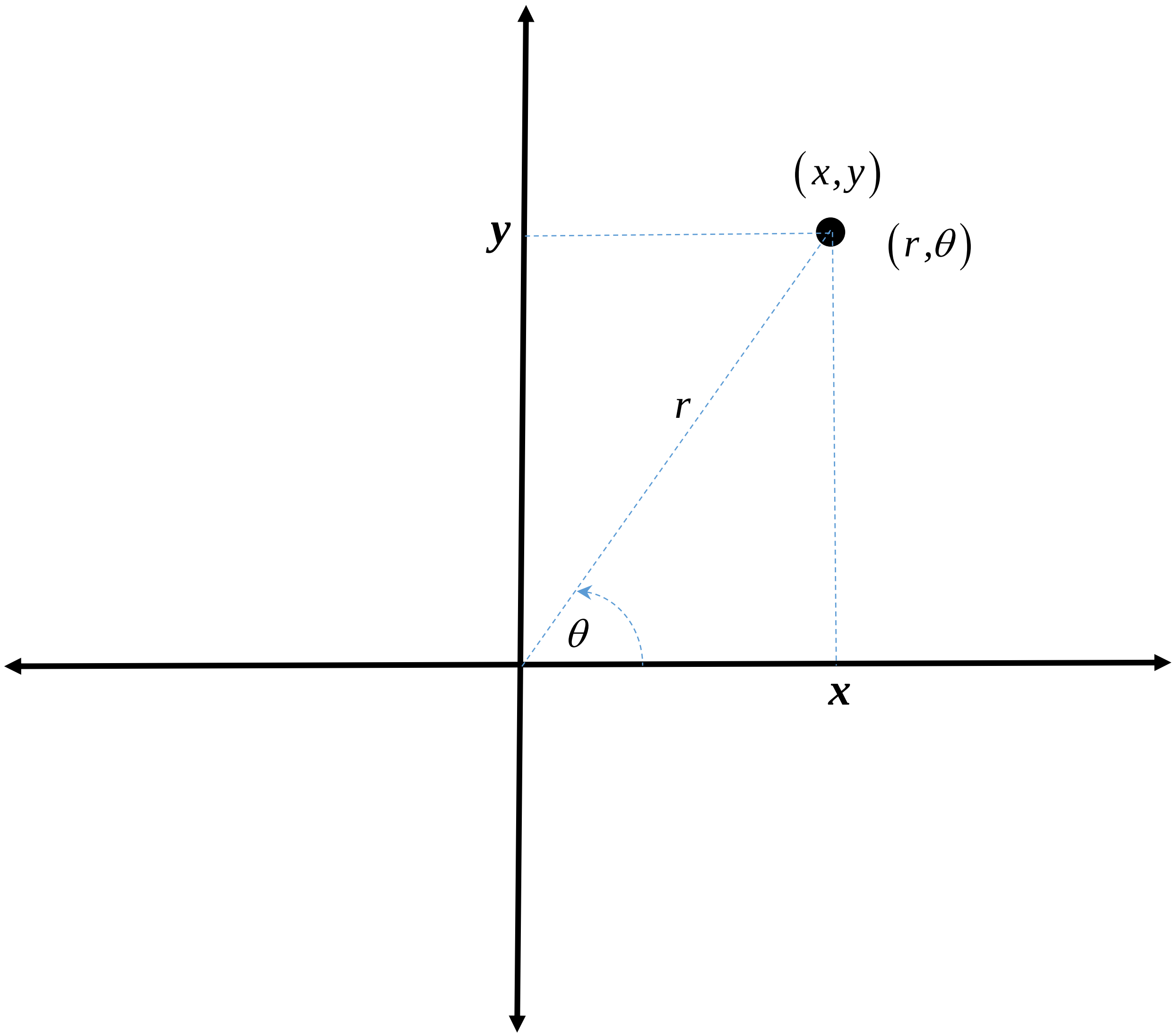
Polar Coordinates:

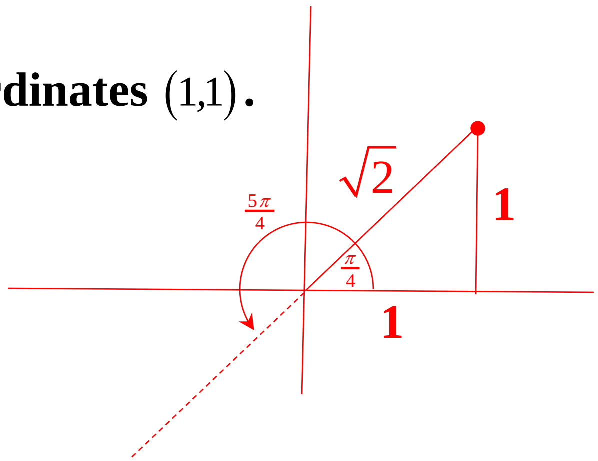
There is an alternative method for locating points in the plane called polar coordinates.



Rectangular/Cartesian coordinates are unique. Polar coordinates are not unique.

Example:

1. Find polar coordinates for the point with rectangular coordinates $(1,1)$.



$$\left(\sqrt{2},\frac{\pi}{4}\right),\left(\sqrt{2},\frac{\pi}{4}\pm 2\pi\right),\left(\sqrt{2},\frac{\pi}{4}\pm 4\pi\right),\ldots$$

$$\left(-\sqrt{2},\frac{5\pi}{4}\right),\left(-\sqrt{2},\frac{5\pi}{4}\pm 2\pi\right),\left(-\sqrt{2},\frac{5\pi}{4}\pm 4\pi\right),\ldots$$

Yes, there are infinitely many polar coordinates for a given pair of rectangular coordinates. Infinitely many angles, and *r* can also be negative.

2. Find polar coordinates for the point with rectangular coordinates $(0,0)$.

$(0,\theta)$, where θ is any angle. **Even the origin has infinitely many polar coordinates.**

Conversion Equations:

$$x^2 + y^2 = r^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\tan\theta = \frac{y}{x}$$

Examples:

1. Find polar coordinates for the rectangular coordinates $\left(-\sqrt{3},1\right)$.

$$x^2 + y^2 = r^2 \Rightarrow \left(-\sqrt{3}\right)^2 + 1^2 = r^2 \Rightarrow r^2 = 4$$

$$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = -\frac{1}{\sqrt{3}}$$

So I'll use $r = 2$ and $\theta = \frac{5\pi}{6}$ to get $\left(2, \frac{5\pi}{6}\right)$.

2. Find rectangular coordinates for the polar coordinates $\left(4, \frac{4\pi}{3}\right)$.

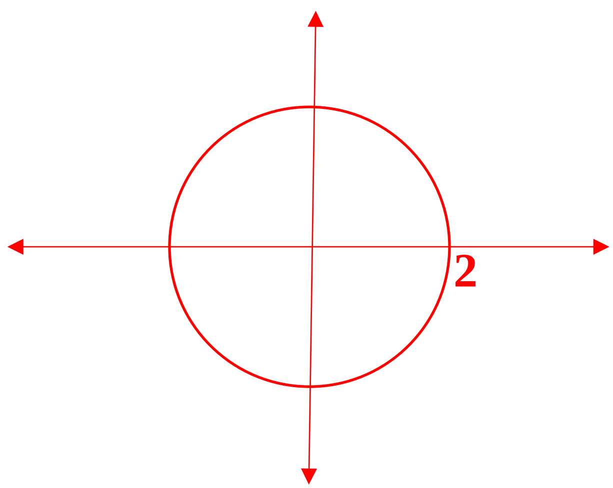
$$x = r\cos\theta \Rightarrow x = 4\cos\frac{4\pi}{3} \Rightarrow x = -2$$

$$y = r\sin\theta \Rightarrow y = 4\sin\frac{4\pi}{3} \Rightarrow y = -2\sqrt{3}$$

So I'll get $\left(-2, -2\sqrt{3}\right)$.

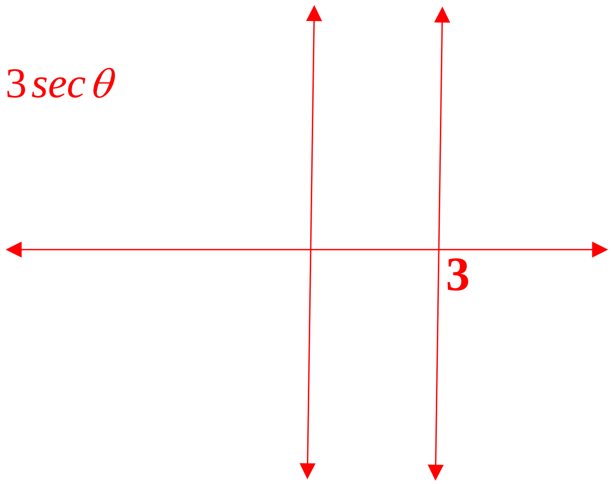
3. Transform the rectangular coordinate equation $x^2 + y^2 = 4$ into an equivalent polar coordinate equation, and graph the solution curve.

$$x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2 \text{ or } r = -2$$



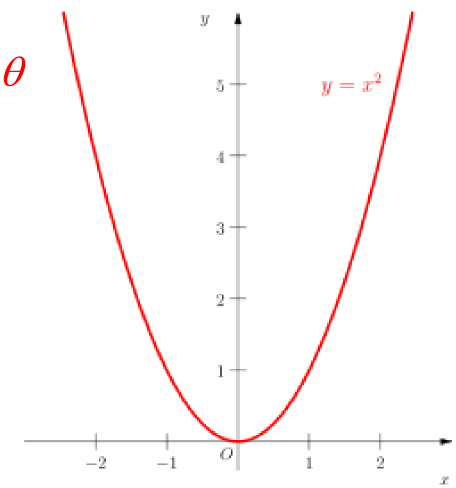
4. Transform the rectangular coordinate equation $x = 3$ into an equivalent polar coordinate equation, and graph the solution curve.

$$x = 3 \Rightarrow r\cos\theta = 3 \Rightarrow r = \frac{3}{\cos\theta} \Rightarrow r = 3\sec\theta$$



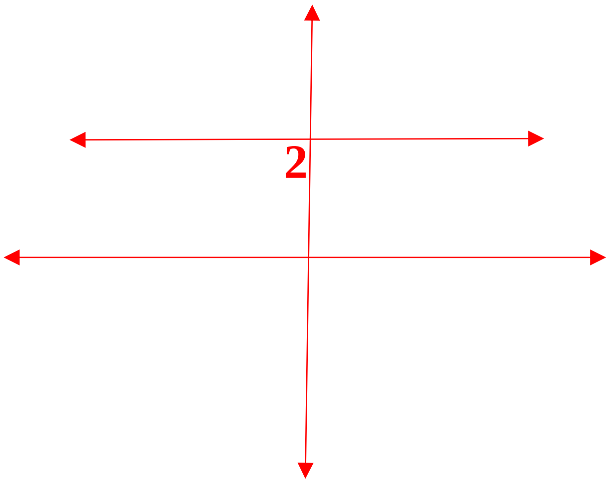
5. Transform the rectangular coordinate equation $y = x^2$ into an equivalent polar coordinate equation, and graph the solution curve.

$$y = x^2 \Rightarrow r\sin\theta = (r\cos\theta)^2 \Rightarrow \sin\theta = r\cos^2\theta \Rightarrow r = \sec\theta\tan\theta$$



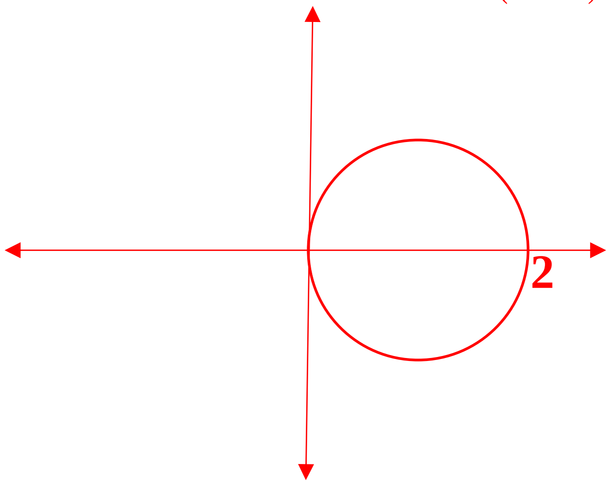
6. Transform the polar coordinate equation $r\sin\theta = 2$ into an equivalent rectangular coordinate equation, and graph the solution curve.

$$r\sin\theta = 2 \Rightarrow y = 2$$



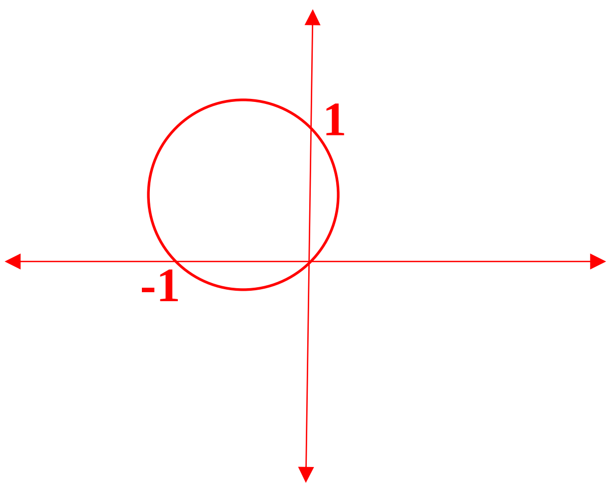
7. Transform the polar coordinate equation $r = 2\cos\theta$ into an equivalent rectangular coordinate equation, and graph the solution curve.

$$r = 2\cos\theta \Rightarrow r^2 = 2r\cos\theta \Rightarrow x^2 + y^2 = 2x \Rightarrow x^2 - 2x + y^2 = 0 \Rightarrow (x-1)^2 + y^2 = 1$$

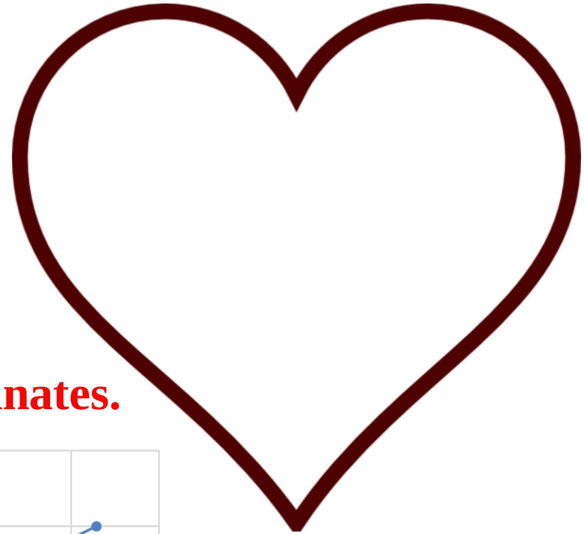


8. Transform the polar coordinate equation $r = sin\theta - cos\theta$ into an equivalent rectangular coordinate equation, and graph the solution curve.

$$r = sin\theta - cos\theta \Rightarrow r^2 = r\,sin\theta - r\,cos\theta \Rightarrow x^2 + y^2 = y - x \Rightarrow x^2 + x + y^2 - y = 0 \Rightarrow \left(x + \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 = \frac{1}{2}$$



Special Polar Coordinate Equations/Graphs:



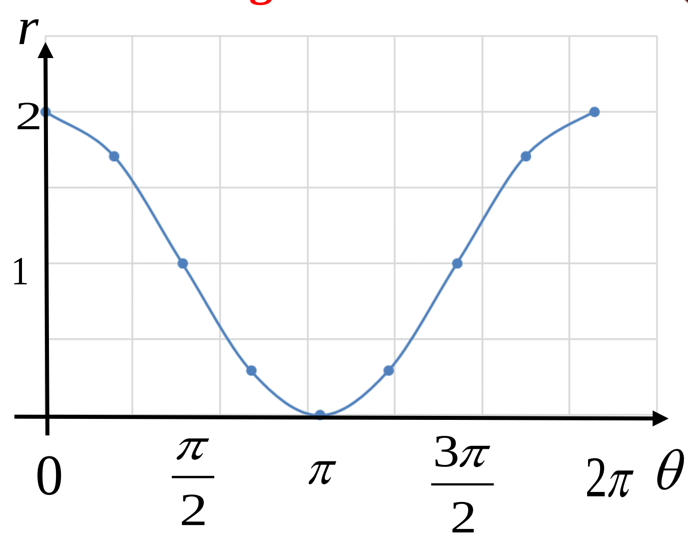
Cardioid:

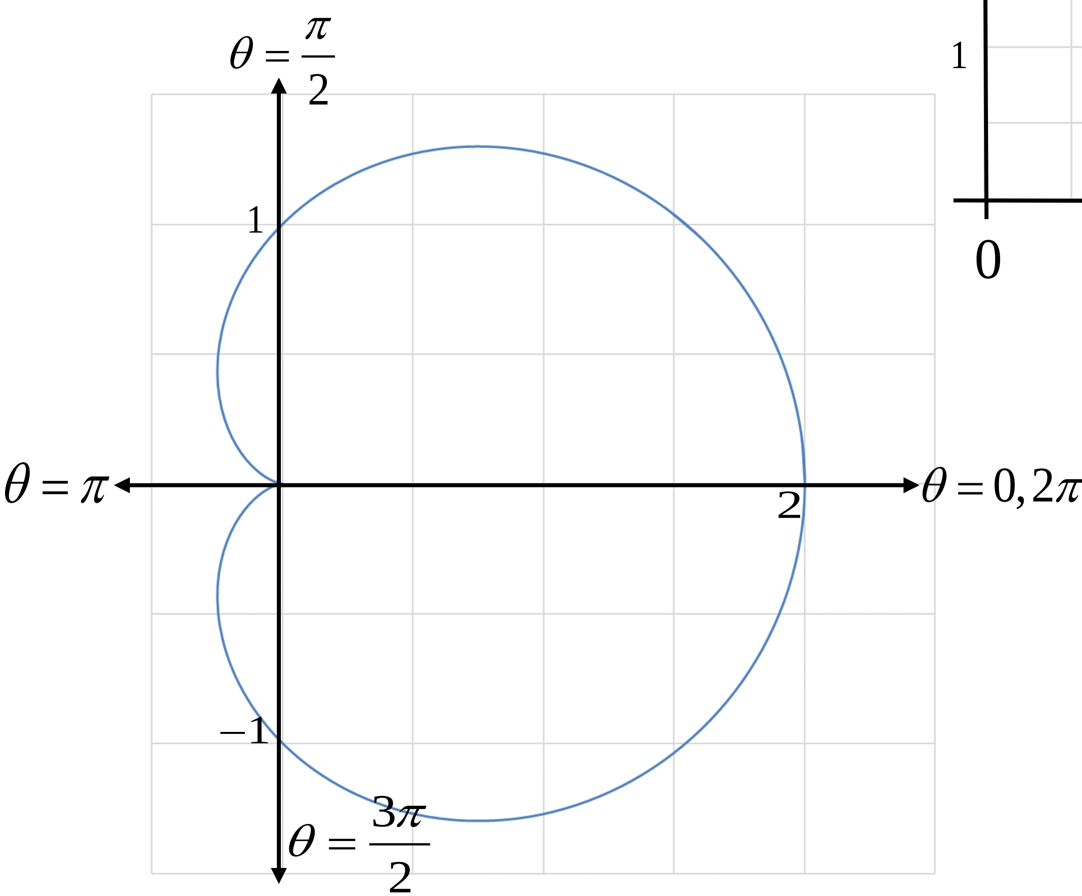
1. $r = 1 + \cos\theta$

Start with an *r* vs. θ plot, treating *r* and θ as rectangular coordinates.

Use the *r* vs. θ plot to help you create

The polar coordinate graph.



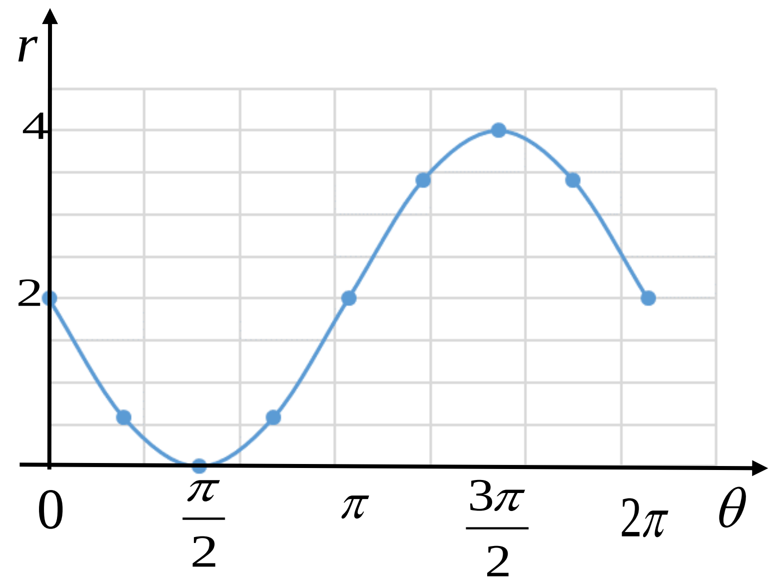


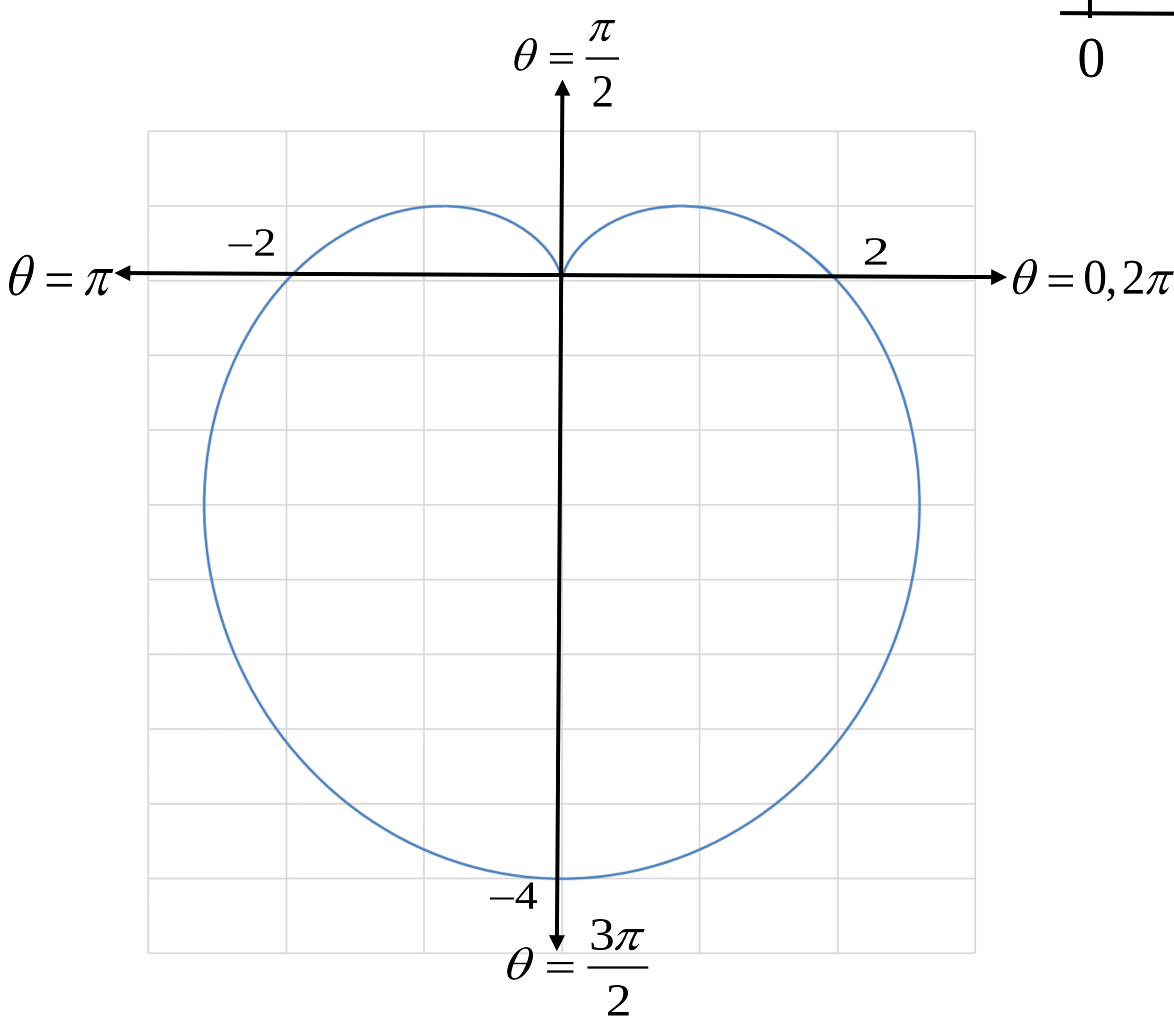
If *r* is zero for a particular value θ_0, then the graph comes into the origin tangent to the line $\theta = \theta_0$.

Here, the value is π.

2. $r = 2(1 - sin\theta)$

To find the angle(s) where *r* is zero, solve the trig. equation, $2(1 - sin\theta) = 0$**.**

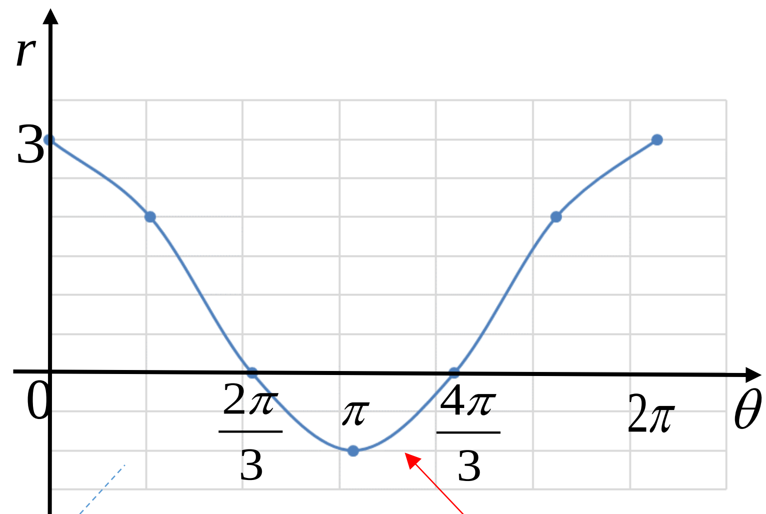


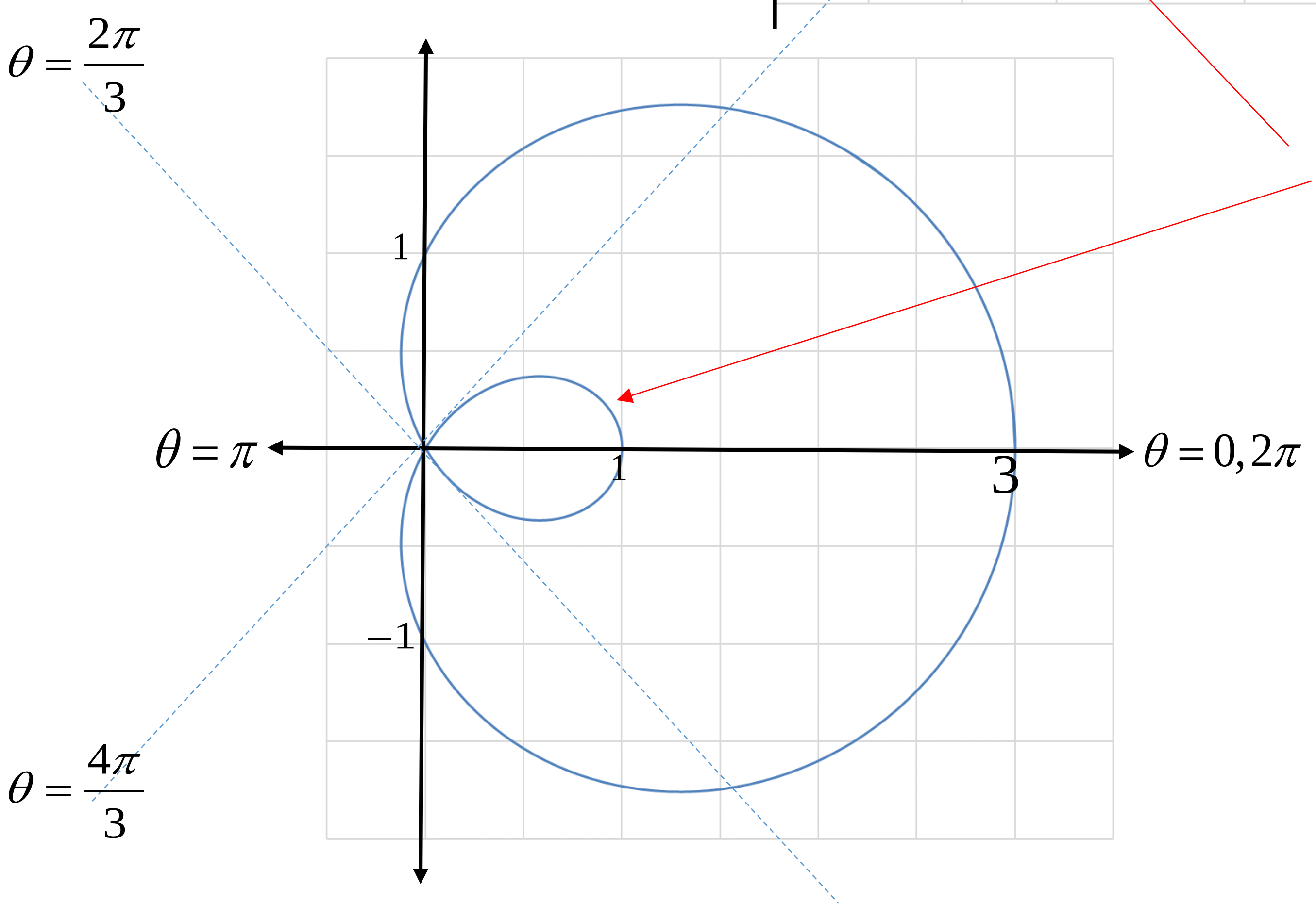


<u>Cardioid with an Inner Loop:</u>

1. $r = 1 + 2\cos\theta$

To find the angle(s) where *r* is zero, solve the trig. equation, $1 + 2\cos\theta = 0$.

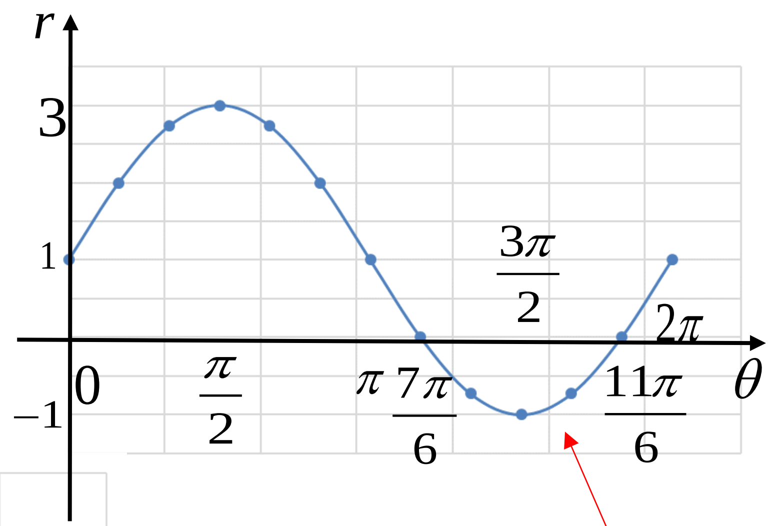


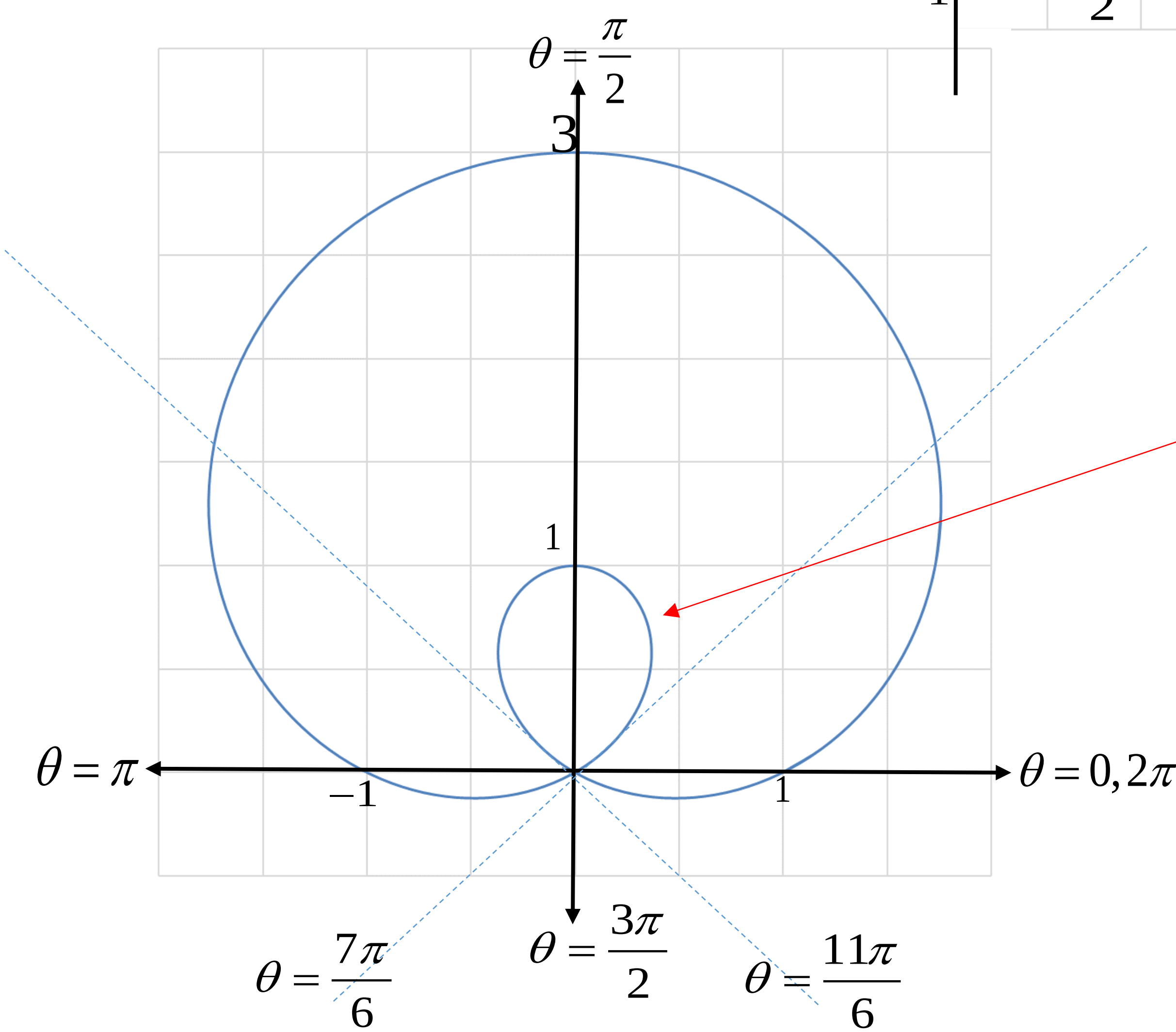


The inner loop is traced when *r* takes on negative values for θ **between** $\frac{2\pi}{3}$ **and** $\frac{4\pi}{3}$**. The graph approaches the origin tangent to these two lines.**

2. $r = 1 + 2\sin\theta$

To find the angle(s) where *r* is zero, solve the trig. equation, $1 + 2\sin\theta = 0$.



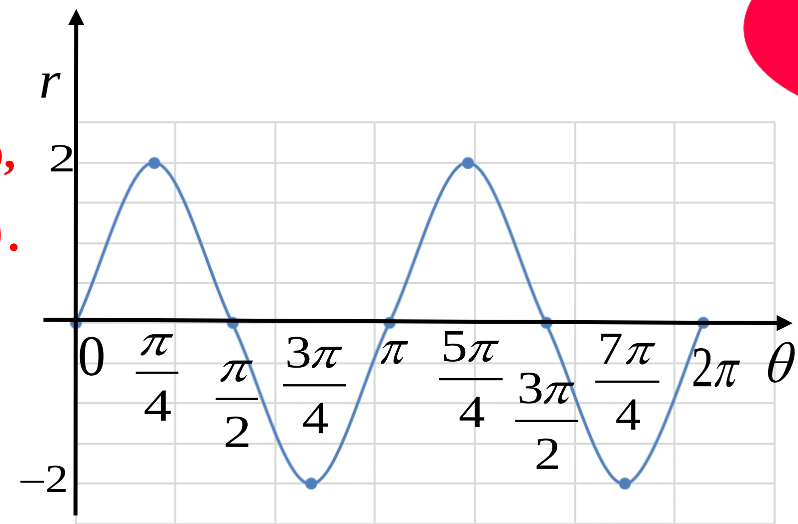


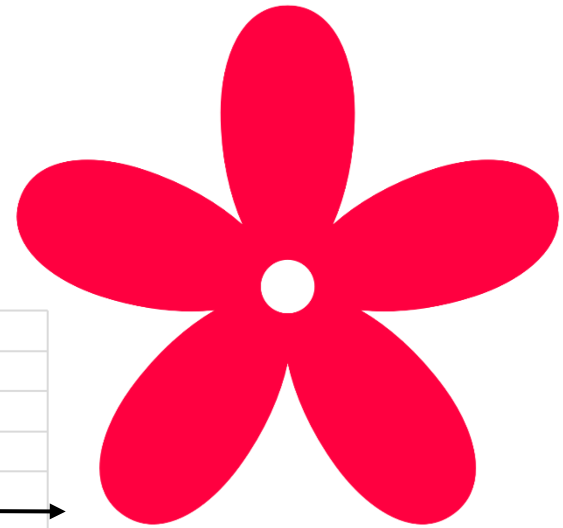
The inner loop is traced when *r* takes on negative values for θ between $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The graph approaches the origin tangent to these two lines.

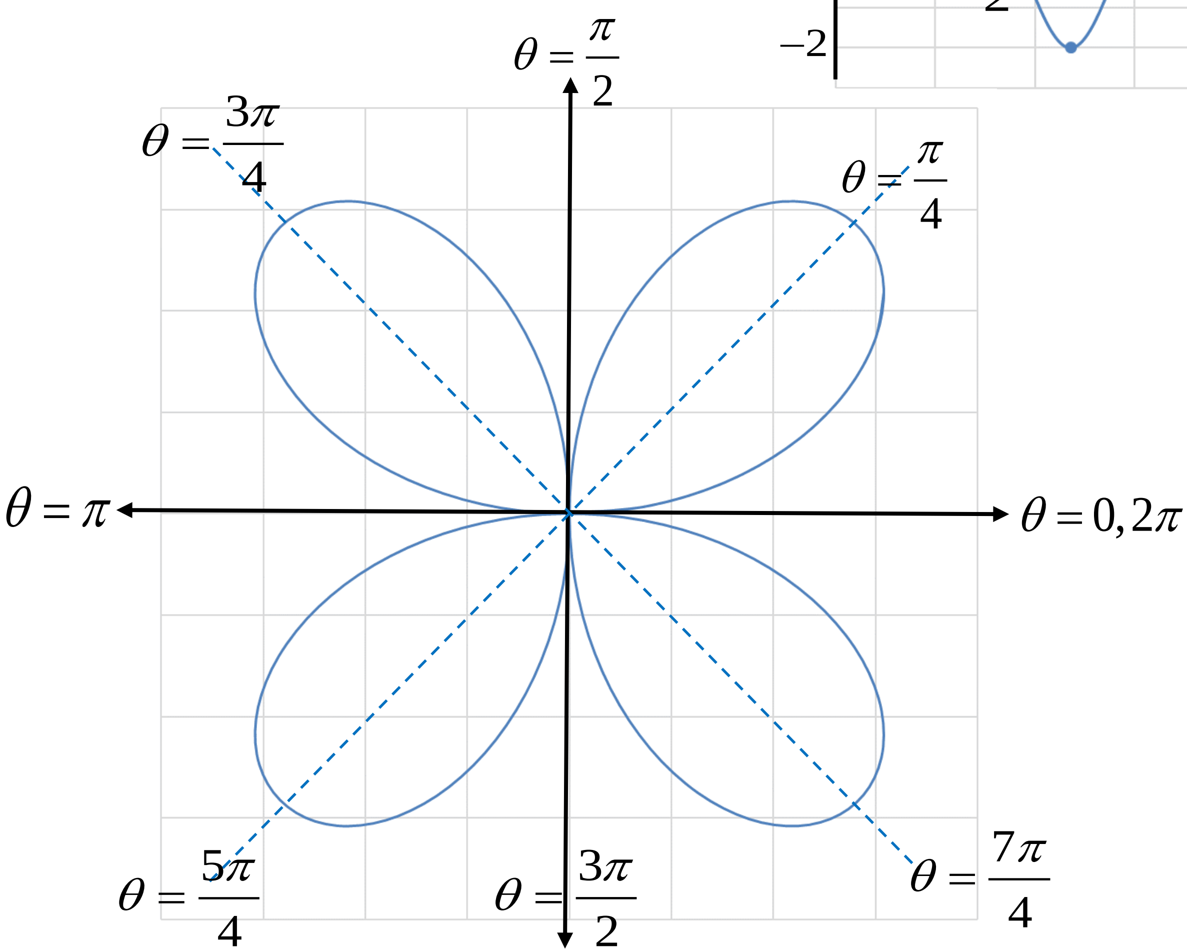
Rose:

$r = 2\sin(2\theta)$

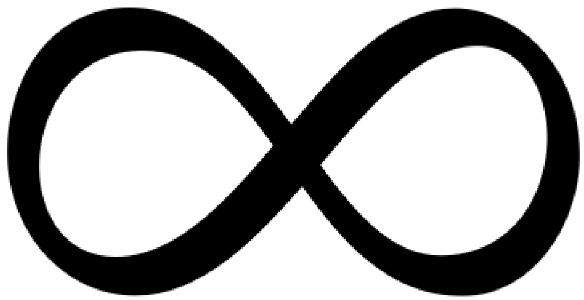
To find the angle(s) where *r* is zero, solve the trig. equation, $2\sin 2\theta = 0$.







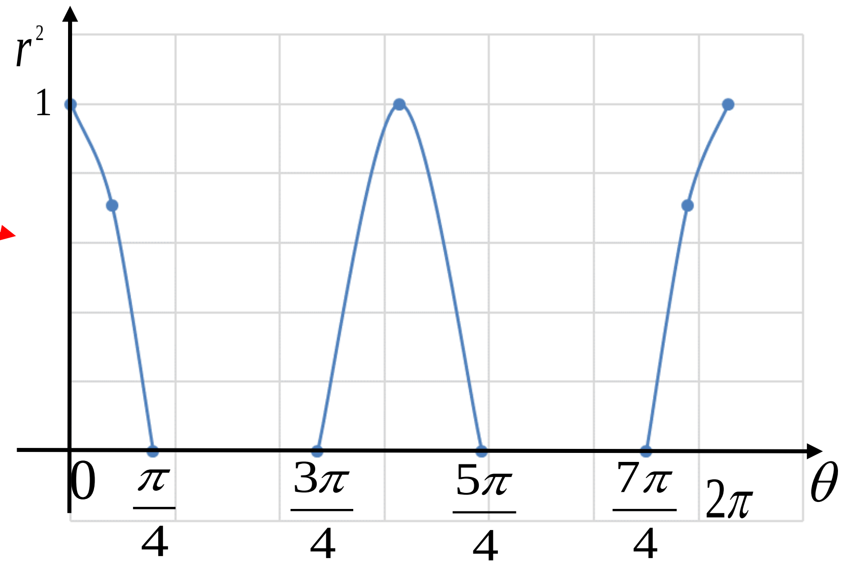
The tips of the four petals occur for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. The graph approaches the origin tangent to the lines corresponding to the angles $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Two of the petals are traced with positive *r* values and two with negative *r* values.

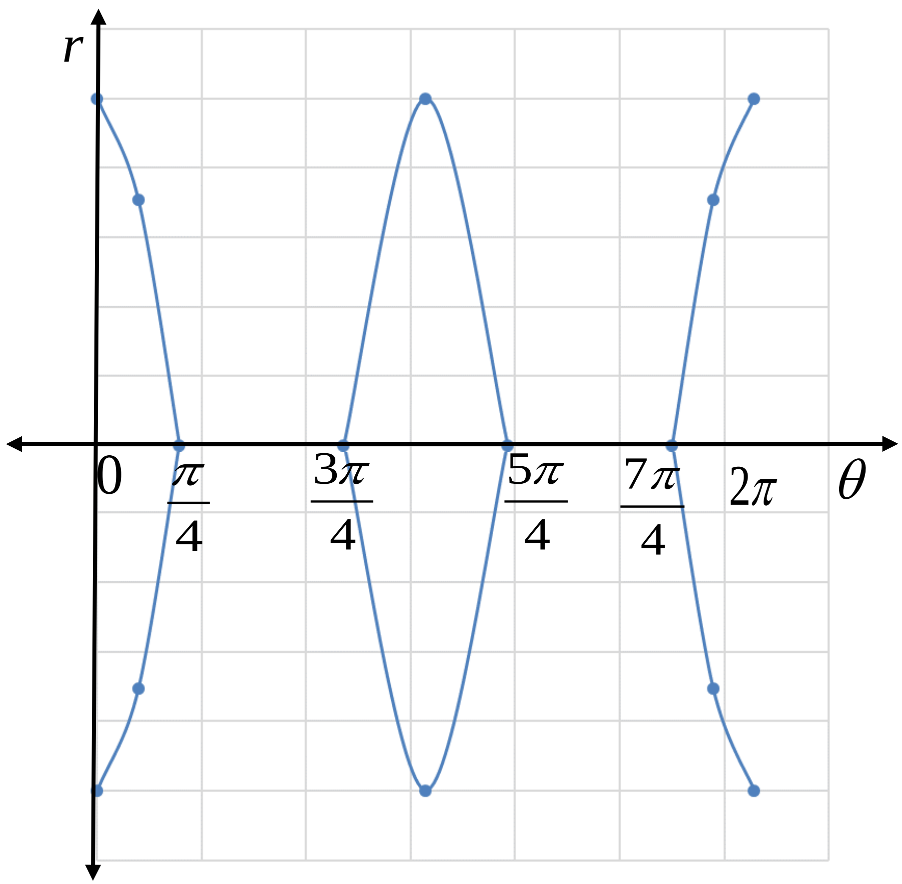


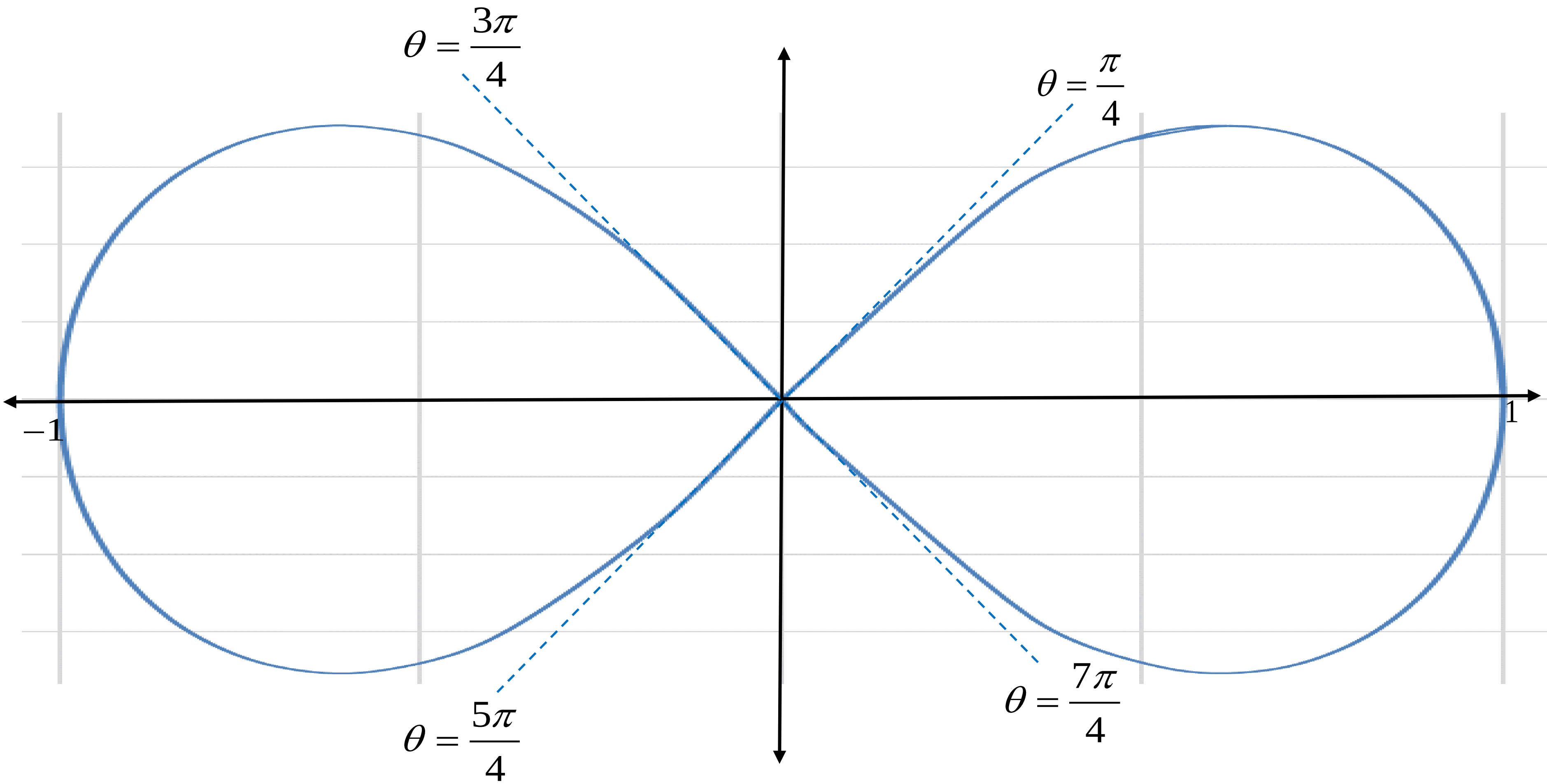
Lemniscate:

$r^2 = cos(2\theta)$

Start with an r^2 vs. θ plot, and then convert it into an r vs. θ plot by taking square-roots. Note that r^2 can't be negative.



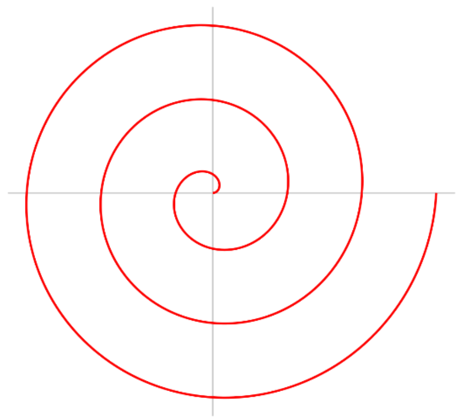


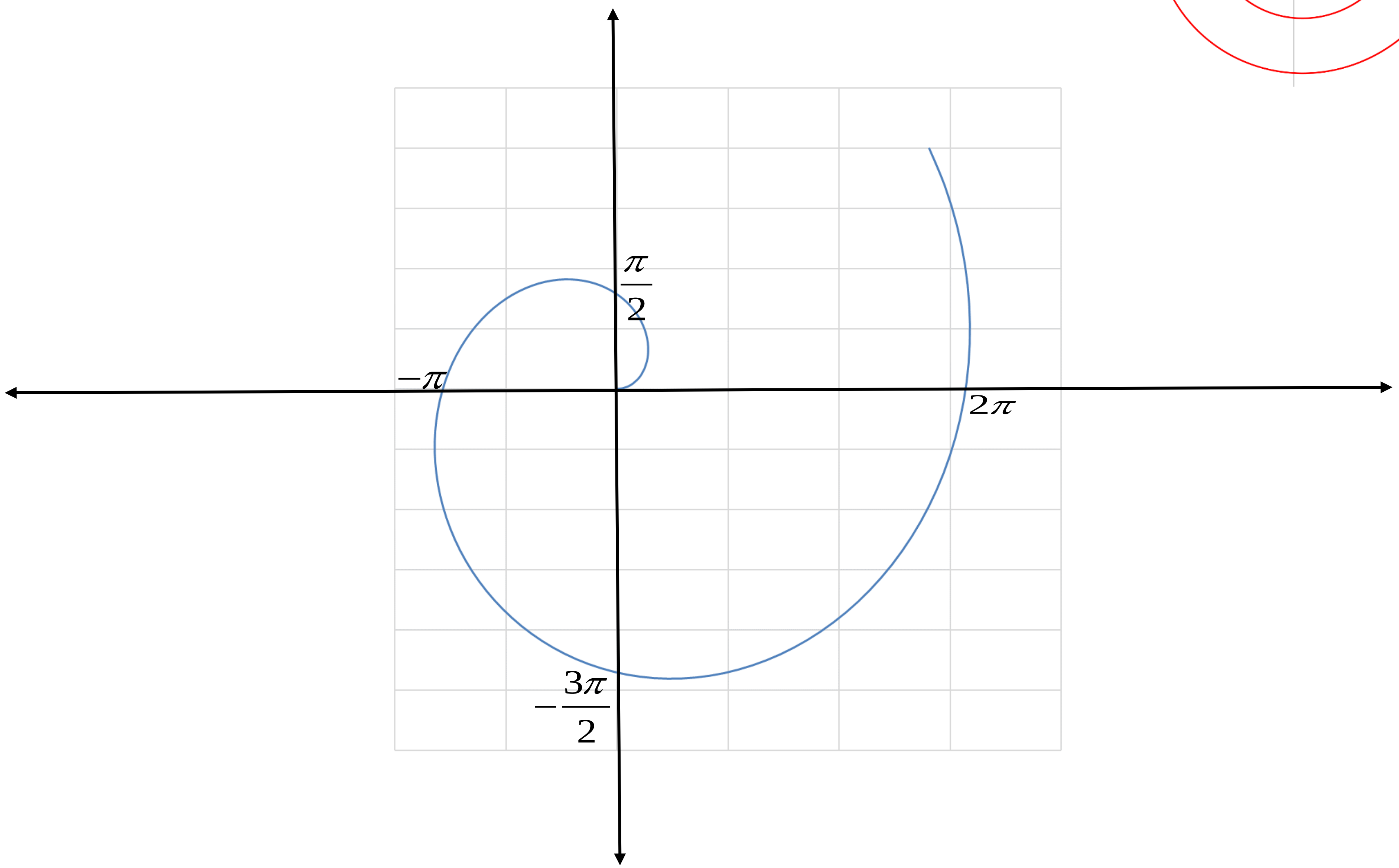


The positive *r* values trace the entire curve once, and the negative *r* values trace it again.

Spiral:

$r = \theta; \theta \geq 0$





Intersections of Polar Graphs:

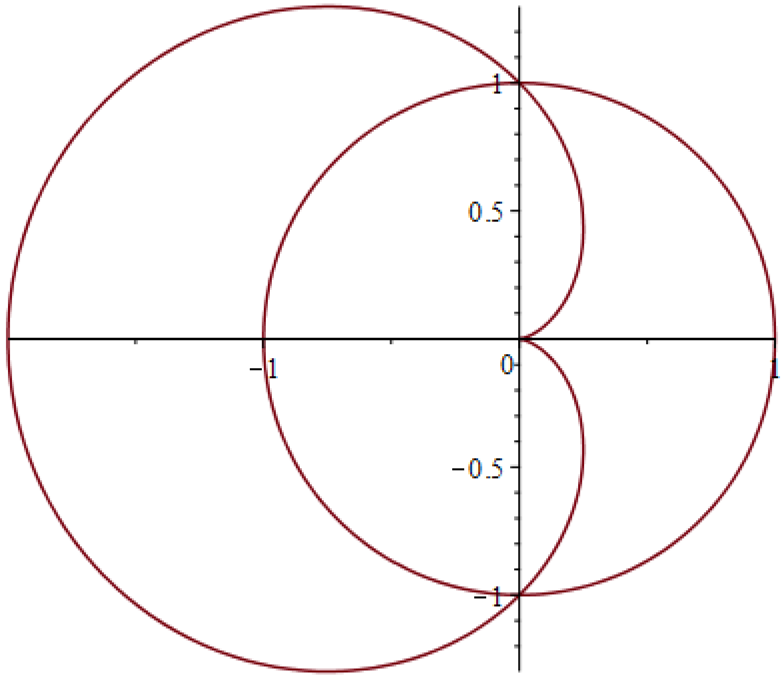
1. Find the points of intersection of the graphs of the polar coordinate equations $r = 1$ and $r = 1 - cos\,\theta$.

Set the equations equal to each other, solve, and use symmetry in the graph, if possible.

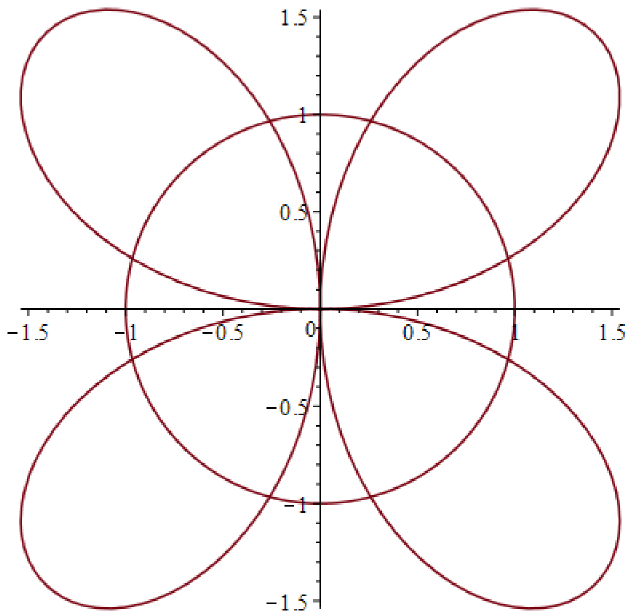
$$1 - \cos\theta = 1 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

So the two points of intersection are

$\left(1, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$.



2. Find the points of intersection of the graphs of the polar coordinate equations $r = 1$ and $r = 2\,sin(2\theta)$.



Set the equations equal to each other, solve, and use symmetry in the graph, if possible.

$$2\sin 2\theta = 1 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{\pi}{6} + 2\pi, \ldots$$

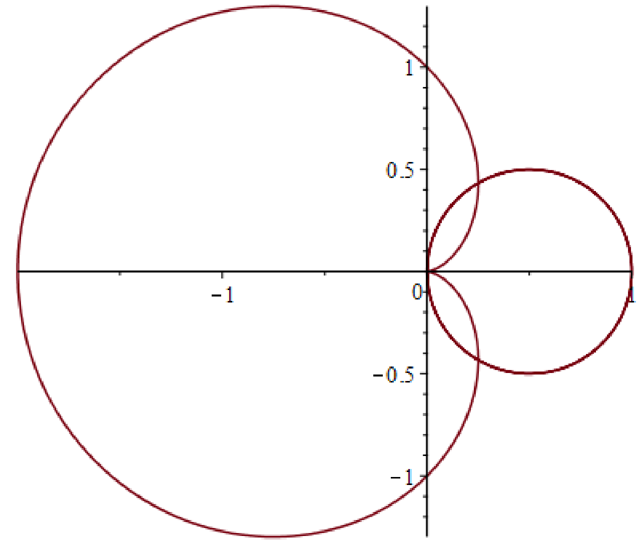
$$\text{or } 2\theta = \frac{5\pi}{6}, \frac{5\pi}{6} + 2\pi, \ldots \Rightarrow \theta = \frac{\pi}{12}, \frac{\pi}{12} + \pi, \ldots$$

$$\text{or } \theta = \frac{5\pi}{12}, \frac{5\pi}{12} + \pi, \ldots$$

So from symmetry, the eight points of intersection are

$$\left(1, \frac{\pi}{12}\right), \left(1, \frac{5\pi}{12}\right), \left(1, \frac{7\pi}{12}\right), \left(1, \frac{11\pi}{12}\right), \left(1, \frac{13\pi}{12}\right), \left(1, \frac{17\pi}{12}\right), \left(1, \frac{19\pi}{12}\right), \left(1, \frac{23\pi}{12}\right).$$

3. Find the points of intersection of the graphs of the polar coordinate equations $r = 1 - \cos\theta$ **and** $r = \cos\theta$**.**



Set the equations equal to each other, solve, and use symmetry in the graph, if possible.

$$\cos\theta = 1 - \cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, \ldots$$

or $\theta = \dfrac{5\pi}{3}, \dfrac{5\pi}{3} + 2\pi, \ldots$

So from the graph, the three points of intersection are

$$(0,0), \left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right).$$

Notice that the origin wasn't detected from setting the equations equal to each other because the curves are at the origin for different values of θ. This is why it's important to look at the graphs of both equations!