Review of Complex Numbers:

The standard form of a complex number is $a+bi$**, where *a* and *b* are real numbers and** $i^2=-1$**.**

Basic Operations:

Addition: $(2+3i)+(5-7i)$

$$(2+3i)+(5-7i)=(2+5)+(3-7)i=\boxed{7-4i}$$

Subtraction: $(-4+3i)-(6-7i)$

$$(-4+3i)-(6-7i)=(-4-6)+(3-(-7))i=\boxed{-10+10i}$$

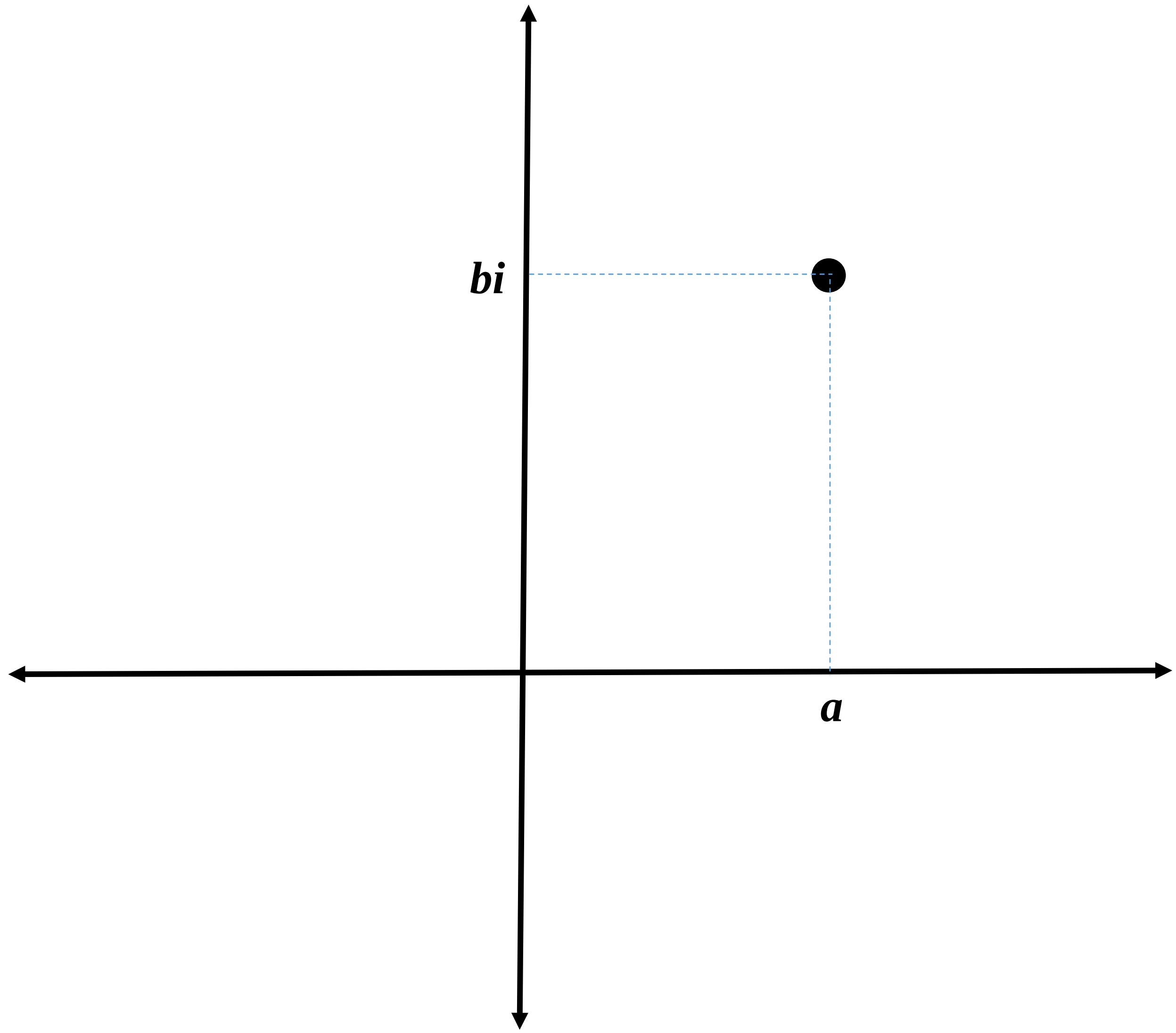
Multiplication: $(2+3i)(5-7i)$

$$(2+3i)(5-7i)=10-14i+15i-21i^2=10+i+21=\boxed{31+i}$$

Division: $(2+3i)\div(1+2i)$

$$(2+3i)\div(1+2i)=\frac{2+3i}{1+2i}\cdot\frac{1-2i}{1-2i}=\frac{2-4i+3i-6i^2}{1-4i^2}=\frac{2-i+6}{5}=\boxed{\frac{8}{5}-\frac{1}{5}i}$$

The standard form is also known as the rectangular form, since the complex number $a + bi$ can be thought of as a point in the complex plane.



The distance that a complex number, $z = a + bi$, is in the complex plane from the origin is called its magnitude or modulus, $|z| = |a + bi| = \sqrt{a^2 + b^2}$.

Example:

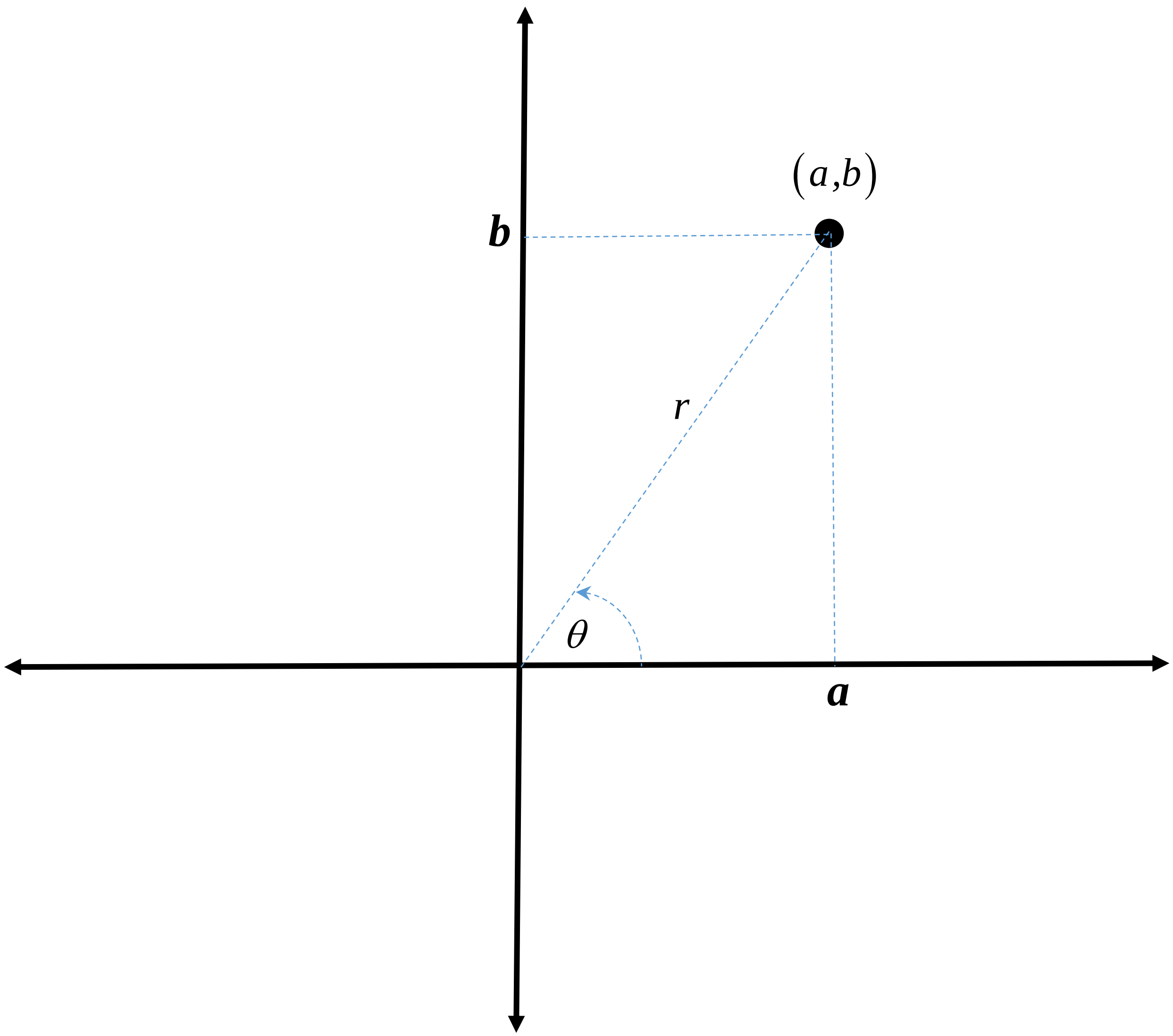
$$|3 - 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

The conjugate or complex conjugate of $a + bi$ is $a - bi$, and the notation is $\overline{a + bi} = a - bi$. For any complex number, *z*, $z\bar{z} = |z|^2$.

Show why.

$$(a + bi)\overline{(a + bi)} = (a + bi)(a - bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |a + bi|^2$$

There is an alternative method for locating and describing complex numbers in the complex plane called polar form-just like polar coordinates.

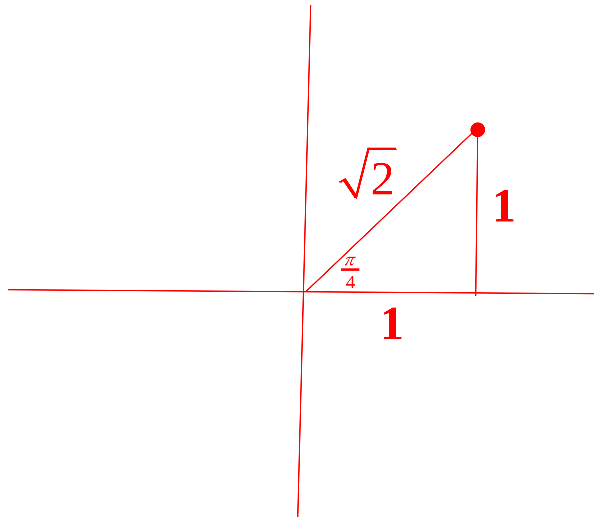


Standard polar form has $r \geq 0$ **,** $0 \leq \theta < 2\pi$**, and is written as** $a+bi = r(\cos\theta + i\sin\theta)$**, where** $r = \sqrt{a^2+b^2}$**.**

Examples:

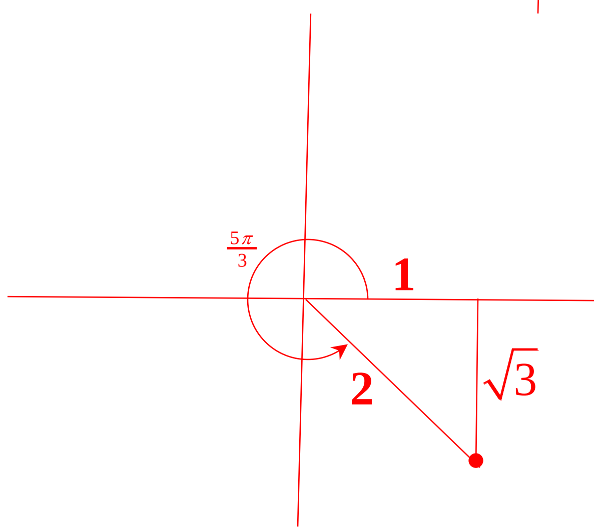
Find the standard polar form of $1+i$**.**

$$1+i = \boxed{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$$



Find the standard polar form of $1-\sqrt{3}i$**.**

$$1-\sqrt{3}i = \boxed{2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)}$$



Products and Quotients of Complex Numbers in Polar Form:

We'll need some trig. identities: $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ **and**

$$\sin(\alpha+\beta)=\cos\alpha\sin\beta+\cos\beta\sin\alpha$$

$$z_1=r_1(\cos\theta_1+i\sin\theta_1) \text{ and } z_2=r_2(\cos\theta_2+i\sin\theta_2)$$

Product: $$z_1z_2=r_1(\cos\theta_1+i\sin\theta_1)\cdot r_2(\cos\theta_2+i\sin\theta_2)$$
$$=r_1r_2\left[(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+i(\cos\theta_1\sin\theta_2+\cos\theta_2\sin\theta_1)\right]$$
$$=r_1r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$$

Similarly,

Quotient: $$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

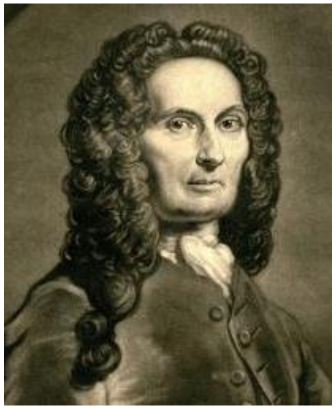
Examples:

$$\left[2\left(cos\frac{5\pi}{12}+i\,sin\frac{5\pi}{12}\right)\right]\cdot\left[3\left(cos\frac{\pi}{12}+i\,sin\frac{\pi}{12}\right)\right]=6\left(cos\frac{6\pi}{12}+i\,sin\frac{6\pi}{12}\right)=6\left(cos\frac{\pi}{2}+i\,sin\frac{\pi}{2}\right)=6i$$

$$\left[2\left(cos\frac{5\pi}{12}+i\,sin\frac{5\pi}{12}\right)\right]\div\left[3\left(cos\frac{\pi}{12}+i\,sin\frac{\pi}{12}\right)\right]=\frac{2}{3}\left(cos\frac{4\pi}{12}+i\,sin\frac{4\pi}{12}\right)$$

$$=\frac{2}{3}\left(cos\frac{\pi}{3}+i\,sin\frac{\pi}{3}\right)$$

$$=\frac{2}{3}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=\frac{1}{3}+\frac{1}{\sqrt{3}}i$$



De Moivre's Theorem:

For $z = r(\cos\theta + i\sin\theta)$ **and *n*, an integer,** $z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$**.**

Examples:

Find $(1+i)^8$**.**

$$(1+i)^8 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^8$$
$$= 16(\cos 2\pi + i\sin 2\pi)$$
$$= 16$$

Find $\left(\sqrt{3}-i\right)^6$**.**

$$\left(\sqrt{3}-i\right)^6 = \left[2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)\right]^6$$
$$= 64(\cos 11\pi + i\sin 11\pi)$$
$$= -64$$

Let's use De Moivre's Theorem to find some roots of imaginary numbers.

Find all the square-roots of *i*.

We want to find a complex number, $z=|z|(\cos\theta+i\sin\theta);0\le\theta<2\pi$**, so that** $z^2=i$**.**

This means that

$$|z|^2(\cos\theta+i\sin\theta)^2=0+i\Rightarrow|z|^2(\cos2\theta+i\sin2\theta)=0+i$$

$$\Rightarrow|z|^2(\cos2\theta+i\sin2\theta)=\cos\left(2n\pi+\frac{\pi}{2}\right)+i\sin\left(2n\pi+\frac{\pi}{2}\right)$$

So $2\theta=2n\pi+\frac{\pi}{2}=\frac{\pi}{2},2\pi+\frac{\pi}{2},4\pi+\frac{\pi}{2},\ldots\Rightarrow\theta=n\pi+\frac{\pi}{4}=\frac{\pi}{4},\pi+\frac{\pi}{4},2\pi+\frac{\pi}{4},\ldots$**, and** $|z|=1$**. The only values of** θ **with** $0\le\theta<2\pi$ **are** $\frac{\pi}{4}$ **and** $\frac{5\pi}{4}$**.**

So the two square-roots of *i* are $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}$ **and** $\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=\frac{-\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}$**.**

Find all the cube-roots of $4\sqrt{3}+4i$**.**

We want to find a complex number, $z=|z|(\cos\theta+i\sin\theta);0\le\theta<2\pi$**, so that** $z^3=4\sqrt{3}+4i$**.**

This means that

$$\left[|z|(\cos\theta+i\sin\theta)\right]^3=4\sqrt{3}+4i\Rightarrow|z|^3(\cos 3\theta+i\sin 3\theta)=4\sqrt{3}+4i$$

$$\Rightarrow|z|^3(\cos 3\theta+i\sin 3\theta)=8\left[\cos\left(2n\pi+\frac{\pi}{6}\right)+i\sin\left(2n\pi+\frac{\pi}{6}\right)\right]$$

So

$$3\theta=2n\pi+\frac{\pi}{6}=\frac{\pi}{6},2\pi+\frac{\pi}{6},4\pi+\frac{\pi}{6},\ldots\Rightarrow\theta=\frac{2n\pi}{3}+\frac{\pi}{18}=\frac{\pi}{18},\frac{2\pi}{3}+\frac{\pi}{18},\frac{4\pi}{3}+\frac{\pi}{18},2\pi+\frac{\pi}{18},\ldots$$

, and the only values of θ **with** $0\le\theta<2\pi$ **are** $\frac{\pi}{18}$**,** $\frac{2\pi}{3}+\frac{\pi}{18}=\frac{13\pi}{18}$**, and** $\frac{4\pi}{3}+\frac{\pi}{18}=\frac{25\pi}{18}$**.**

We also need to have that $|z|=2$**. So the three cube-roots of** $4\sqrt{3}+4i$ **are** $2\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)$**,** $2\left(\cos\frac{13\pi}{18}+i\sin\frac{13\pi}{18}\right)$**, and** $2\left(\cos\frac{25\pi}{18}+i\sin\frac{25\pi}{18}\right)$**.**