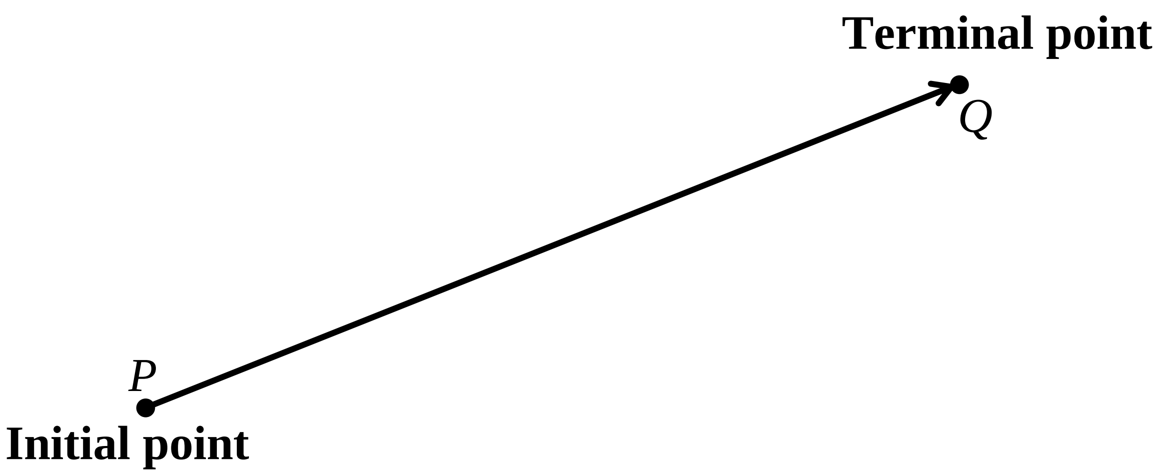
Vectors:

For us, vectors are directed line segments(arrows). They represent quantities that have both a direction and a magnitude(amount).

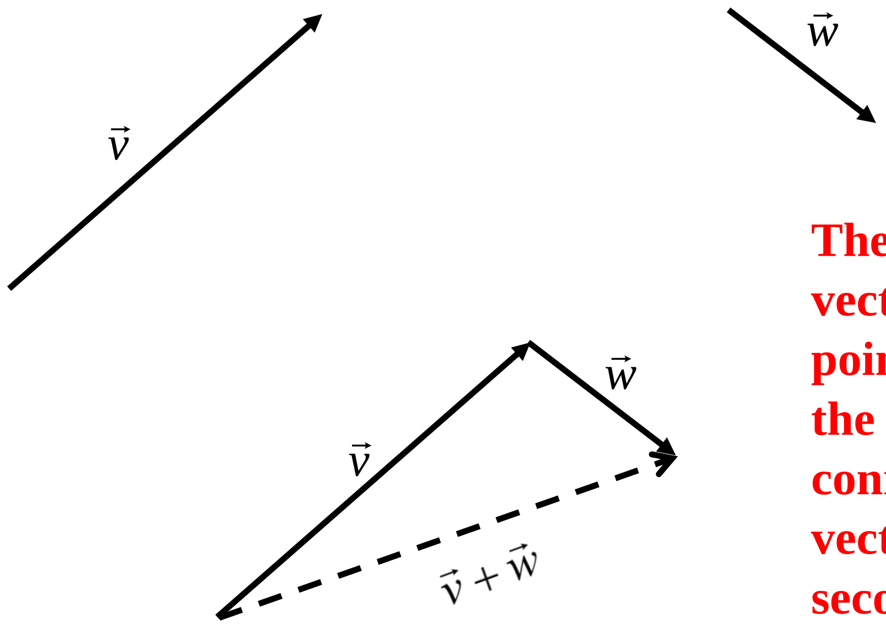


$\overrightarrow{PQ}$ or $\vec{v}$

The length of a vector is its magnitude, and the direction can be interpreted as an angle or slope of the arrow. There is a special vector whose initial and terminal points are the same, i.e. its magnitude is zero. This vector is called the zero vector, $\vec{0}$.

The Sum of Two Vectors:

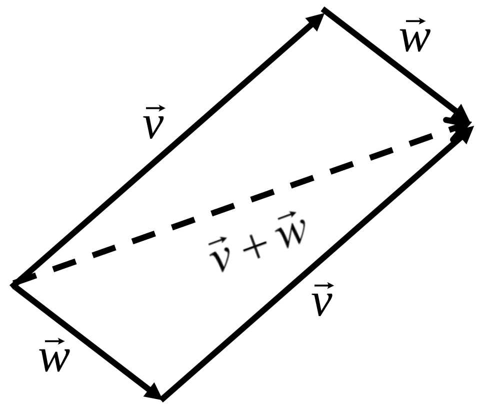
$\vec{v} + \vec{w}$



The head(terminal point) of the first vector is attached to the tail(initial point) of the second vector. The sum of the two vectors is the vector that connects the initial point of the first vector to the terminal point of the second vector.

Properties of Vector Addition:

1. $\vec{v}+\vec{w}=\vec{w}+\vec{v}$**, Commutative Property.**



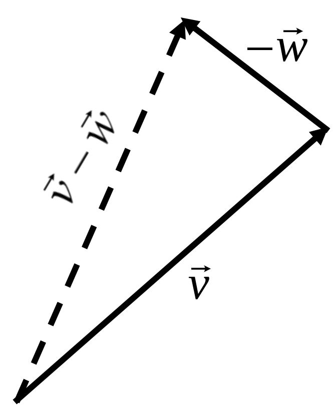
Whether you start with $\vec{v}$ or $\vec{w}$, the sum will be the same. The sum in either order corresponds to a diagonal of the pictured parallelogram.

2. $\vec{u}+\left(\vec{v}+\vec{w}\right)=\left(\vec{u}+\vec{v}\right)+\vec{w}$**, Associative Property**

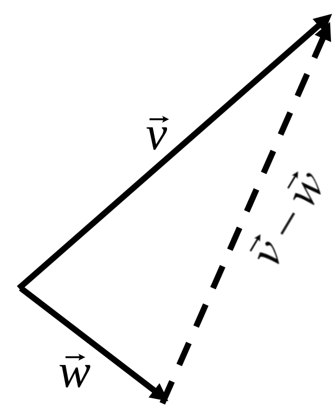
3. $\vec{v}+\vec{0}=\vec{0}+\vec{v}=\vec{v}$**, Zero Vector Property**

4. $\vec{v}-\vec{w}=\vec{v}+(-\vec{w})$**, where** $-\vec{w}$ **is a vector with same magnitude as** $\vec{w}$ **but the opposite direction. Vector Subtraction.**

Vector subtraction can be done by converting the subtraction into addition:
$\vec{v}-\vec{w}=\vec{v}+(-\vec{w})$



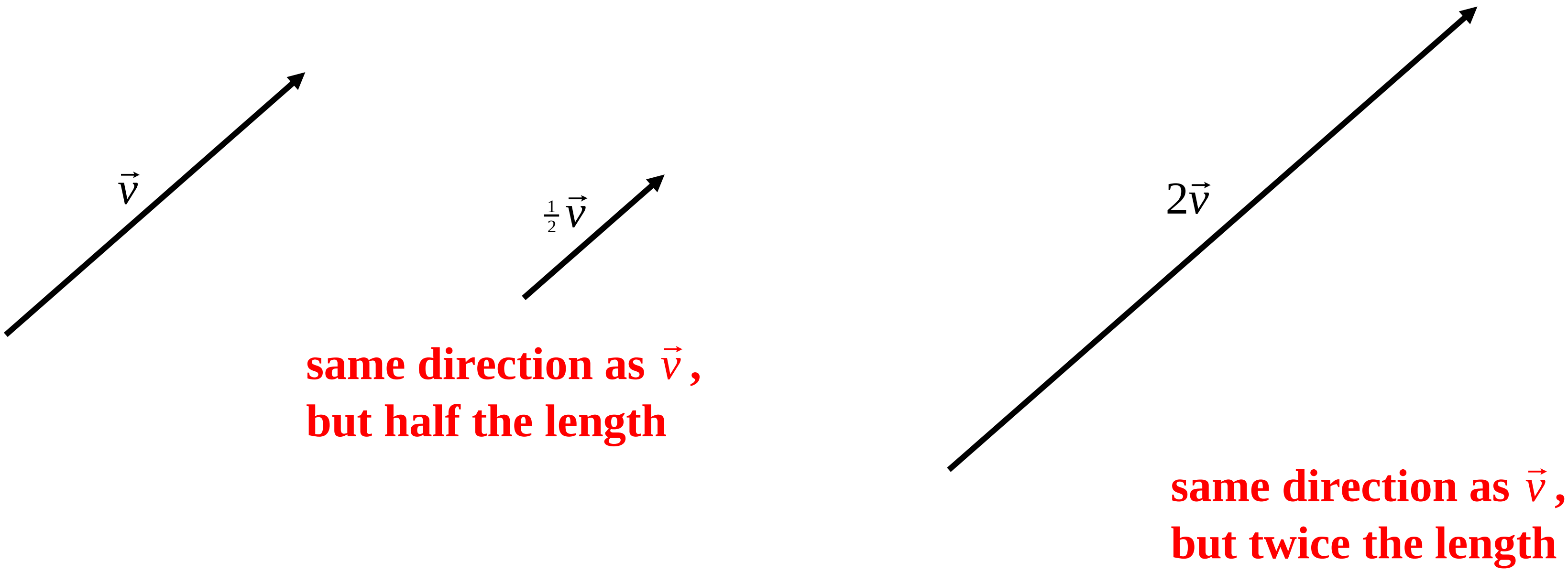
Or, for vector subtraction, you can connect the two vectors at their tails. The vector that connects the terminal point of the second vector to the intial point of the first vector is the difference of the two vectors.



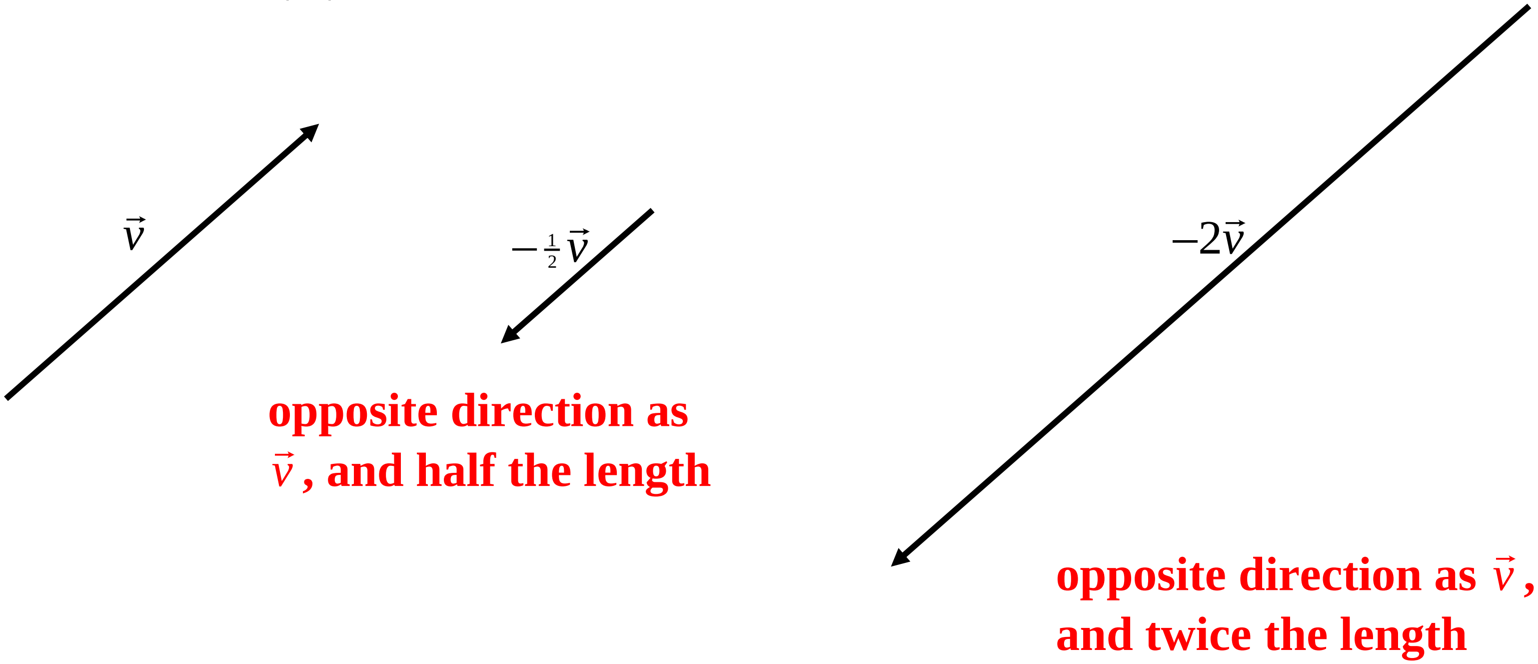
Multiplying Vectors by Numbers:

The process is called scalar multiplication.

If $\alpha > 0$, then the vector $\alpha\vec{v}$ is the vector with the same direction as $\vec{v}$, but whose magnitude is α times the magnitude of $\vec{v}$.



If $\alpha < 0$, then the vector $\alpha\vec{v}$ is the vector with the opposite direction as $\vec{v}$, but whose magnitude is $|\alpha|$ times the magnitude of $\vec{v}$.



If $\alpha = 0$ or $\vec{v} = \vec{0}$ then $\alpha\vec{v} = \vec{0}$.

Properties of Scalar Multiplication:

1. $0\vec{v} = \vec{0}$

2. $1\vec{v} = \vec{v}$

3. $-1\vec{v} = -\vec{v}$

4. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

5. $\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$

6. $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$

Magnitude of a Vector:

The abbreviation for the magnitude of the vector $\vec{v}$, is $\|\vec{v}\|$.

Properties of Magnitude:

1. $\|\vec{v}\| \geq 0$

2. $\|\vec{v}\| = 0$ **if and only if** $\vec{v} = \vec{0}$

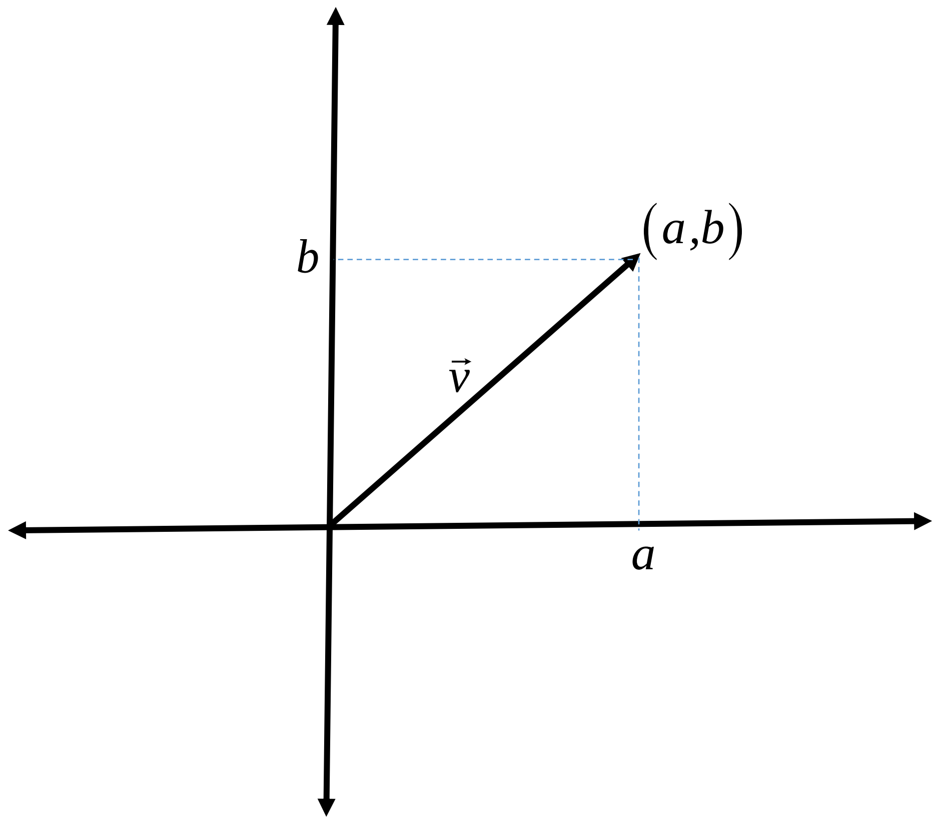
3. $\|-\vec{v}\| = \|\vec{v}\|$

4. $\|\alpha\vec{v}\| = |\alpha|\|\vec{v}\|$

If $\|\vec{v}\| = 1$, then $\vec{v}$ is called a unit vector.

Algebraic or Component Form of Vectors:

Place the initial point of the vector $\vec{v}$ at the origin.



The component form of the vector $\vec{v}$ is $\langle a,b \rangle$, i.e. $\vec{v} = \langle a,b \rangle$. In general, if you know the coordinates of the initial point, (x_1, y_1), and terminal point, (x_2, y_2), of $\vec{v}$, then $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

***a* is called the horizontal component, and *b* is called the vertical component.**

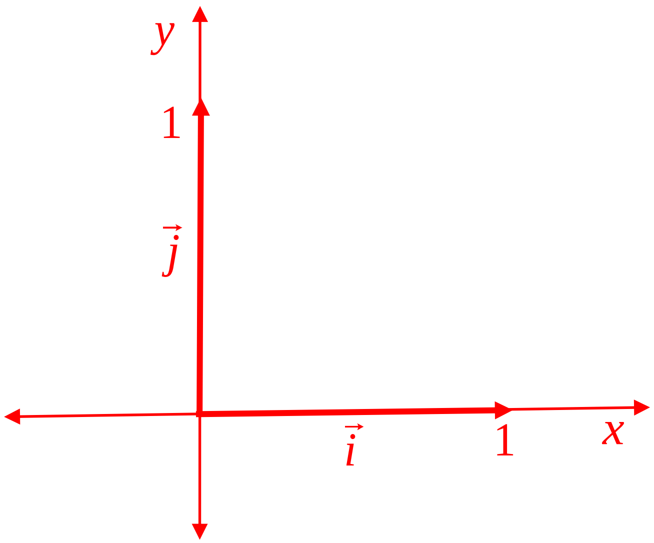
Example:

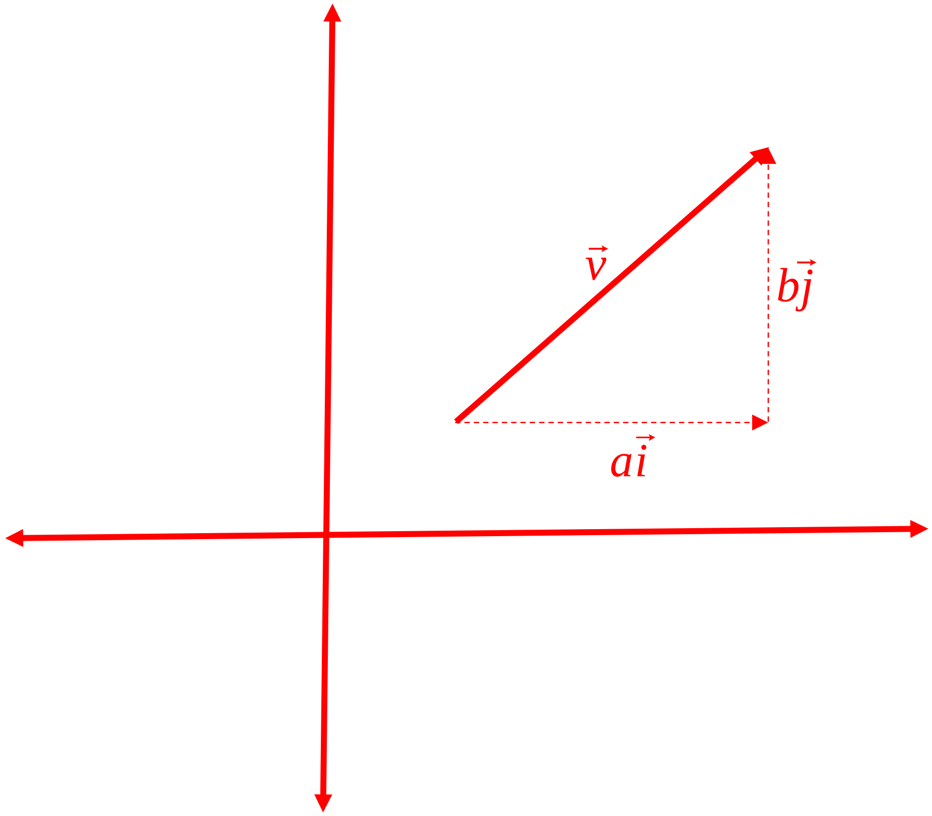
Find the component form of the vector that connects the point $(1,-2)$ to the point $(-3,4)$.

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle -3-1, 4-(-2) \rangle = \boxed{\langle -4,6 \rangle}$$

There is an alternative method of representing the component form of a vector using the special unit vectors $\vec{i}$ and $\vec{j}$.

$\vec{i}$ is a unit vector pointing in the same direction as the positive *x*-axis, and $\vec{j}$ is a unit vector pointing in the same direction as the positive *y*-axis.





So if $\vec{v} = \langle a, b \rangle$, then $\vec{v} = a\vec{i} + b\vec{j}$.

Vector Operations Using Algebraic Vectors:

If $\vec{v} = \langle a_1, b_1 \rangle$ **and** $\vec{w} = \langle a_2, b_2 \rangle$ **then**

1. $\vec{v} + \vec{w} = \langle a_1 + a_2, b_1 + b_2 \rangle$

2. $\vec{v} - \vec{w} = \langle a_1 - a_2, b_1 - b_2 \rangle$

3. $\alpha\vec{v} = \langle \alpha a_1, \alpha b_1 \rangle$

4. $\|\vec{v}\| = \sqrt{a_1^2 + b_1^2}$

Operations with vectors are much easier using the idea of the component form or algebraic form of the vectors. You don't have to draw pictures or rely on geometry.

Examples:

If $\vec{v} = \langle 2,-3 \rangle$ **and** $\vec{w} = \langle 4,5 \rangle$ **then find**

$$\vec{v} + \vec{w} = \langle 6,2 \rangle$$

$$\vec{v} - \vec{w} = \langle -2,-8 \rangle$$

$$5\vec{v} = \langle 10,-15 \rangle$$

$$3\vec{v} - 2\vec{w} = \langle 6,-9 \rangle - \langle 8,10 \rangle = \boxed{\langle -2,-19 \rangle}$$

$$\|\vec{v}\| = \sqrt{2^2 + (-3)^2} = \boxed{\sqrt{13}}$$

<u>Finding Unit Vectors with the Same Direction:</u>

If $\vec{v}$ **is a non-zero vector, then the vector** $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$ **is a unit vector with same direction as** $\vec{v}$**.**

Show why.

Since $\frac{1}{\|\vec{v}\|}$ **is a positive number, then** $\vec{u}$ **must have the same direction as** $\vec{v}$**. Now let's show that the length of** $\vec{u}$ **is equal to 1.**

$$\|\vec{u}\| = \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \frac{1}{\|\vec{v}\|} \cdot \|\vec{v}\| = 1$$

Example:

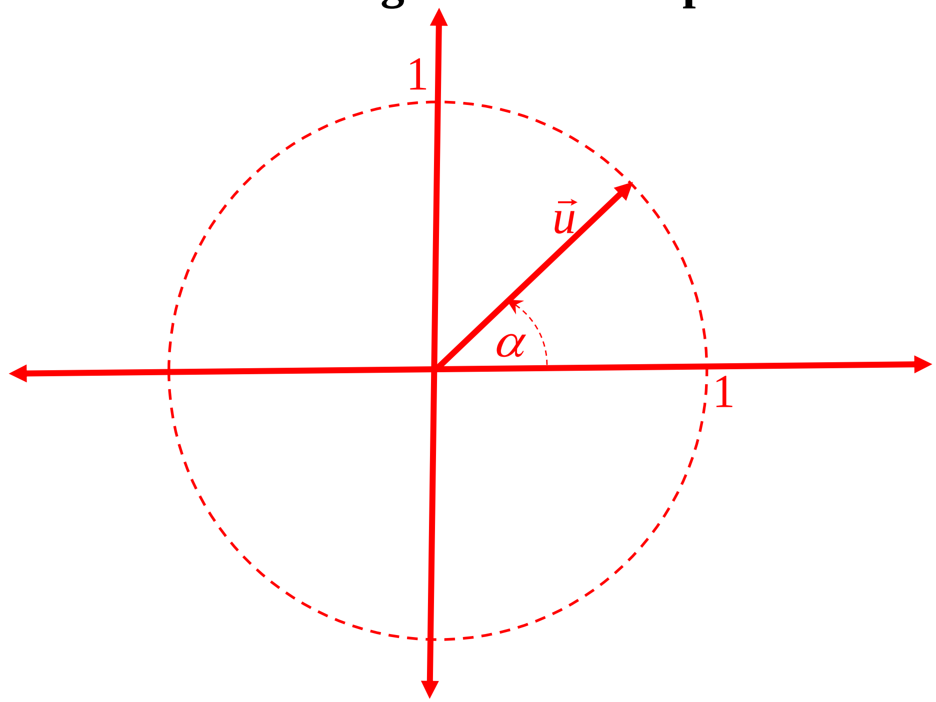
Find a unit vector with the same direction as the vector $\vec{v} = \langle 2, -3 \rangle$**.**

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 2,-3 \rangle}{\sqrt{2^2 + (-3)^2}} = \frac{\langle 2,-3 \rangle}{\sqrt{13}} = \boxed{\left\langle \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right\rangle}$$

If $\vec{u}$ is a unit vector with the same direction as $\vec{v}$, then $\vec{v} = \|\vec{v}\|\vec{u}$.

This is true because $\|\vec{v}\|\vec{u}$ has the same direction and length as $\vec{v}$.

If you place the initial point of a unit vector, $\vec{u}$, at the origin, its terminal point will lie on the unit circle. It will determine an angle α with the positive *x*-axis with $0 \le \alpha < 2\pi$.



α is called the direction angle of the unit vector $\vec{u}$, and

$\vec{u} = \langle \cos\alpha, \sin\alpha \rangle = \cos\alpha\vec{i} + \sin\alpha\vec{j}$.

So if you are given the magnitude $\|\vec{v}\|$ of a non-zero vector $\vec{v}$ and a direction angle α, then $\vec{v} = \|\vec{v}\|\langle \cos\alpha, \sin\alpha \rangle = \underbrace{\|\vec{v}\|\left(\cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}\right)}_{\text{polar form of the vector}}$**.**

Examples:

1. Find the direction angle of the vector $\vec{v} = \langle 1, -\sqrt{3} \rangle$**.**

$$\|\vec{v}\| = 2 \Rightarrow \vec{v} = 2\left\langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \right\rangle \Rightarrow \cos\alpha = \frac{1}{2}, \sin\alpha = \frac{-\sqrt{3}}{2} \Rightarrow \boxed{\alpha = \frac{5\pi}{3}}$$

2. Find the vector of magnitude 6 and direction angle of $\frac{5\pi}{6}$**.**

$$\vec{v} = \|\vec{v}\|\langle \cos\alpha, \sin\alpha \rangle = 6\left\langle \cos\frac{5\pi}{6}, \sin\frac{5\pi}{6} \right\rangle = 6\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = \boxed{\left\langle -3\sqrt{3}, 3 \right\rangle}$$