Dot Product of Two Vectors:

If $\vec{v} = \langle a_1, b_1 \rangle$ **and** $\vec{w} = \langle a_2, b_2 \rangle$**, then the dot product of the two vectors is** $\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2$**.**

The dot product of two vectors will always produce a scalar(number).

Examples:

1. $\langle 1,2 \rangle \cdot \langle 2,-3 \rangle$

$$\langle 1,2 \rangle \cdot \langle 2,-3 \rangle = (1)(2) + (2)(-3)$$
$$= 2 - 6 = \boxed{-4}$$

2. $\langle 1,2 \rangle \cdot \langle 2,-1 \rangle$

$$\langle 1,2 \rangle \cdot \langle 2,-1 \rangle = (1)(2) + (2)(-1)$$
$$= 2 - 2 = \boxed{0}$$

3. $\langle 1,2 \rangle \cdot \langle 1,2 \rangle$

$$\langle 1,2 \rangle \cdot \langle 1,2 \rangle = (1)(2) + (1)(2)$$
$$= 2 + 2 = \boxed{4}$$

4. $\langle 1,2 \rangle \cdot \langle 0,0 \rangle$

$$\langle 1,2 \rangle \cdot \langle 0,0 \rangle = (1)(0) + (2)(0)$$
$$= 0 + 0 = \boxed{0}$$

Properties of Dot Product:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

Prove it.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 = v_1 u_1 + v_2 u_2 = \vec{v} \cdot \vec{u}$$

2. $\vec{u} \cdot \left(\vec{v} + \vec{w} \right) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

Prove it.

$$\begin{aligned} \vec{u} \cdot \left(\vec{v} + \vec{w} \right) = \left\langle u_1, u_2 \right\rangle \cdot \left\langle v_1 + w_1, v_2 + w_2 \right\rangle &= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 \\ &= \left(u_1 v_1 + u_2 v_2 \right) + \left(u_1 w_1 + u_2 w_2 \right) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \end{aligned}$$

3. $\vec{u} \cdot \vec{u} = \left\| \vec{u} \right\|^2$

Prove it.

$$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 = \left(\sqrt{u_1^2 + u_2^2} \right)^2 = \left\| \vec{u} \right\|^2$$

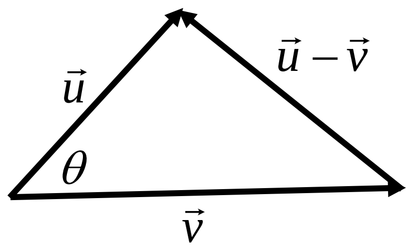
4. $\vec{u} \cdot \vec{0} = \vec{0} \cdot \vec{u} = 0$

Prove it.

$$\vec{u} \cdot \vec{0} = \left(u_1 \right)\left(0 \right) + \left(u_2 \right)\left(0 \right) = 0 + 0 = 0$$

What does the dot product represent?

If $\vec{u}$ and $\vec{v}$ are non-zero vectors, then we can form the following triangle.



From the Law of Cosines, we get that

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|cos\,\theta$$

So

$$(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\|\vec{u}\|\|\vec{v}\|cos\,\theta$$

expanding the left-side

$$\Rightarrow \vec{u}\cdot\vec{u} - 2\vec{u}\vec{v} + \vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\|\vec{u}\|\|\vec{v}\|cos\,\theta$$

cancelling common terms on both sides

$$\Rightarrow -2\vec{u}\cdot\vec{v} = -2\|\vec{u}\|\|\vec{v}\|cos\,\theta$$

$$\Rightarrow \boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|cos\,\theta}$$

So the dot product of the vectors $\vec{u}$ and $\vec{v}$ is the product of their lengths and the cosine of the angle between the two vectors.

For $\vec{u}$ and $\vec{v}$ non-zero vectors, we can find the angle between the vectors, θ, with $0 \le \theta \le \pi$, from the formula

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

Example:

Find the angle between the vectors $\left\langle \sqrt{3},1 \right\rangle$ and $\left\langle -2\sqrt{3},2 \right\rangle$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-6+2}{(2)(4)} = -\frac{1}{2} \Rightarrow \boxed{\theta = \frac{2\pi}{3}}$$

Parallel Vectors:

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if one of them is a scalar multiple of the other, i.e. $\vec{u} = \alpha\vec{v}$. If $\alpha > 0$, then the angle between the vectors is 0. If $\alpha < 0$, then the angle between the vectors is π. The zero vector is considered to be parallel to all vectors.

Orthogonal Vectors:

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are orthogonal(perpendicular) if the angle between them is $\frac{\pi}{2}$. In this case, $\vec{u} \cdot \vec{v} = 0$. The zero vector is considered to be orthogonal to every vector, so two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

Examples:

Determine if the given pair of vectors are parallel, orthogonal, or neither.

1. $\langle 1,2\rangle$ **and** $\langle 3,6\rangle$

$\langle 3,6\rangle = 3\langle 1,2\rangle \Rightarrow \boxed{\text{parallel}}$

2. $\langle 1,2\rangle$ **and** $\left\langle -\frac{1}{3},-\frac{2}{3}\right\rangle$

$\langle 1,2\rangle = -3\left\langle -\frac{1}{3},-\frac{2}{3}\right\rangle \Rightarrow \boxed{\text{parallel}}$

3. $\langle 1,2\rangle$ **and** $\langle -2,1\rangle$

$\langle 1,2\rangle \cdot \langle -2,1\rangle = -2+2=0 \Rightarrow \boxed{\text{orthogonal}}$

4. $\langle 1,2\rangle$ **and** $\langle 4,-7\rangle$

$\langle 1,2\rangle \cdot \langle 4,-7\rangle = 4-14=-10$

$\Rightarrow$ not orthogonal

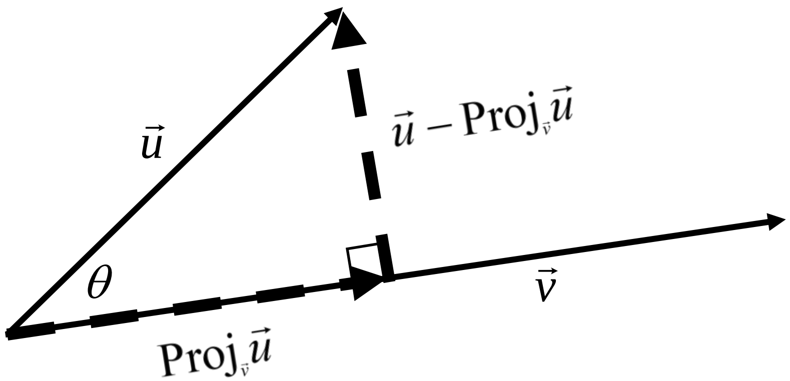
They're not multiples of each other

$\Rightarrow$ not parallel, so $\boxed{\text{neither}}$

Vector Projection:

Given two non-zero vectors $\vec{u}$ and $\vec{v}$, we would like to write the vector $\vec{u}$ as a sum of two vectors: one parallel to $\vec{v}$ and one orthogonal to $\vec{v}$.

The vector parallel to $\vec{v}$ is called the projection of $\vec{u}$ onto $\vec{v}$ and is denoted by $\text{Proj}_{\vec{v}}\,\vec{u}$. The vector orthogonal to $\vec{v}$ is $\vec{u}-\text{Proj}_{\vec{v}}\,\vec{u}$.



$$\vec{u}=\text{Proj}_{\vec{v}}\vec{u}+\left(\vec{u}-\text{Proj}_{\vec{v}}\vec{u}\right)$$

$$\text{Proj}_{\vec{v}}\vec{u}=\|\vec{u}\|\cos\theta\frac{\vec{v}}{\|\vec{v}\|}=\left(\frac{\|\vec{u}\|\|\vec{v}\|\cos\theta}{\|\vec{v}\|^{2}}\right)\vec{v}=\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$\vec{u} = \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}}_{\text{parallel to } \vec{v}} + \underbrace{\left(\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}\right)}_{\text{orthogonal to } \vec{v}}$$

Example:

Write the vector $\vec{u} = \langle 2,3 \rangle$ as a sum of two vectors: one parallel to $\vec{v} = \langle 1,-3 \rangle$ and one orthogonal to $\vec{v} = \langle 1,-3 \rangle$.

$$\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} + \left(\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}\right) = \left(\frac{2-9}{10}\right)\langle 1,-3 \rangle + \left(\langle 2,3 \rangle - \left(\frac{2-9}{10}\right)\langle 1,-3 \rangle\right)$$

$$= \underbrace{\left\langle -\frac{7}{10}, \frac{21}{10} \right\rangle}_{\text{parallel to } \vec{v}} + \underbrace{\left\langle \frac{27}{10}, \frac{9}{10} \right\rangle}_{\text{orthogonal to } \vec{v}}$$